DOI:10.13878/j.cnki.jnuist.2022.05.014

范景威[1]　周伟灿[2,3]　冯也骋[2]　官元红[1]

台风强度变化最快时涡度的解析解

摘要

利用旋转坐标系中的基本方程推导得到动能方程,通过动能的局地变化率定义刻画台风强度变化率的能量泛函,对泛函作变分得到 Euler-Lagrange 方程.分析方程可知,当台风强度变化率达到最大时,摩擦力、气压梯度力、重力和动能梯度满足四力平衡关系.因此,这四个力确定的向量可以作为台风强度的预报因子,更准确地确定台风系统强度变化率达到最大的时间点.进一步通过风场变分分解提取到有旋场中的最大涡旋,得到台风强度变化最快时涡度和流场的解析解,对研究台风发展过程中速度的变化趋势和台风的空间结构具有实用价值,为台风路径和强度预报提供了一定的理论指引.

关键词

台风强度;涡度;解析解;变分法;风场分解

中图分类号　O175.2;P444

文献标志码　A

收稿日期　2021-09-19

资助项目　国家自然科学基金(41475091,41975087)

作者简介

范景威,男,硕士生,研究方向为偏微分方程及其应用.jingweifan1997@139.com

周伟灿(通信作者),男,博士,教授,研究方向为微分方程在大气科学中的应用.zhwnim@nuist.edu.cn

1 南京信息工程大学 数学与统计学院,南京,210044

2 南京信息工程大学 气象灾害教育部重点实验室/气候与环境变化国际合作联合实验室/气象灾害预报预警与评估协同创新中心,南京,210044

3 宿迁泽达职业技术学院 宿迁,223800

0　引言

台风强度变化是台风研究的一个重要课题,是当前台风观测和预报中的重难点与前沿问题.影响台风强度变化的因子可以归结为以下三类:环境气流、下垫面和台风内部结构[1-2].随着监测手段的发展以及数值模式理论和技术的不断成熟,国内外学者针对台风强度变化的研究取得了一定成果,已有研究内容涉及诸多方面,包括环境风垂直切变、冷空气、高低空急流、西风槽、地形以及海面温度和喷沫等[3-9].

变分法是研究泛函极值的数学方法,其主要思想是构造适当的泛函,对泛函作变分并取变分为零,研究此时满足的函数性态.变分法广泛地应用于力学、数学物理反问题、气象资料同化和大气运动的稳定性问题中.Arnold[10]和 Vallis[11]运用变分原理,指出流体运动和地转平衡分别是在位涡守恒及势能守恒的约束下,使总能量达到极小.伍荣生[12]利用 Finlayson[13]提出的限制性变分原理,求得了与非线性涡度方程对应的 Lagrange 函数的近似表达式.Barth 等[14]从线性浅水方程和系统能量变分出发,推导了从高程和深度平均速度中去除惯性重力波的方法,指出变分滤波器在提前降低重力波方面的优势.黄思训等[15]通过二次变分方法研究台风流场结构,将实测风场分解成无旋场和有旋场,再将有旋流场分解成对称涡旋和非对称涡旋对,两次提取到的都是最大涡旋.Wang 等[16]用变分法推导了赤道电离层 Rayleigh-Taylor(RT)稳定和不稳定的充分条件,并通过系统特征值计算了 RT 稳定和不稳定区域.Badin 等[17]将变分原理应用于波动动力学中,得到地表水波方程以及非线性问题的波频散关系.近年来,随着变分同化技术的发展与成熟,它被广泛地应用于大气和海洋等不同领域[18-21],其目的是充分利用气象四维观测资料,在动力预报模式中通过变分得到大气海洋最优状态估计,以获得更好的模式初始场及预报效果.

目前对台风强度的研究多采用统计、诊断分析、数值模拟和敏感性试验等方法,缺少理论层面的研究.此外,前人的研究多采用直接分解的方法将台风流场分解成涡旋流场和无旋流场[15],这种分解无法得到有旋场中的最大涡旋.本文通过能量泛函变分,研究台风强度变化率最大时台风能量满足的关系,在四力平衡关系的基础上提取到有旋流场中的最大涡旋,给出台风强度变化最快时涡度的解析解.该结果为预报方程提供了一定的理论指引,对研究台风发展过程中速度的变化趋势和台风的层次结构具有实用价值.

1 能量泛函变分

假设大气是均匀不可压缩的流体，由连续方程有

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1}$$

其中，$u=u(x,y,z,t)$ 为纬向风速，$v=v(x,y,z,t)$ 为经向风速，$w=w(x,y,z,t)$ 为垂直风速.

大气运动遵循牛顿第二运动定律.对于惯性坐标系，牛顿第二运动定律可表示为

$$\frac{\mathrm{d_a}\boldsymbol{V}_\mathrm{a}}{\mathrm{d}t}=\sum_i \boldsymbol{F}_i, \tag{2}$$

其中，$\boldsymbol{F}_i$ 是作用于单位气块上的外力，$\frac{\mathrm{d_a}}{\mathrm{d}t}$ 为在惯性坐标系中的全微商，$\boldsymbol{V}_\mathrm{a}$ 为绝对速度，$\frac{\mathrm{d_a}\boldsymbol{V}_\mathrm{a}}{\mathrm{d}t}$ 为绝对加速度.绝对加速度与相对加速度满足

$$\frac{\mathrm{d_a}\boldsymbol{V}_\mathrm{a}}{\mathrm{d}t}=\frac{\mathrm{d_r}\boldsymbol{V}_3}{\mathrm{d}t}+2\boldsymbol{\Omega}\times\boldsymbol{V}_3-\boldsymbol{\Omega}^2\boldsymbol{R}, \tag{3}$$

其中，$\boldsymbol{V}_3=(u,v,w)$ 为三维速度矢量，$\boldsymbol{\Omega}$ 为地球自转角速度矢量，$\boldsymbol{R}$ 为气块所在的纬圈平面内从地轴到该气块的距离矢量，其大小为 $\boldsymbol{R}=\boldsymbol{r}\cos\phi$.作用于空气微团的气压梯度力、分子黏性力、湍流黏性力和重力的表达式分别为

$$\boldsymbol{F}_1=-\frac{1}{\rho}\nabla_3 p, \tag{4}$$

$$\boldsymbol{F}_2=\frac{\mu}{\rho}\nabla_3^2\boldsymbol{V}_3, \tag{5}$$

$$\boldsymbol{F}_3=\frac{1}{\rho}\frac{\partial}{\partial z}\left(A_z\frac{\partial \boldsymbol{V}}{\partial z}\right), \tag{6}$$

$$\boldsymbol{g}=-\frac{GM}{r^3}\boldsymbol{r}+\boldsymbol{\Omega}^2\boldsymbol{R}, \tag{7}$$

其中，ρ 为密度，$p=p(x,y,z,t)$ 为气压，∇_3 为三维微分算子，μ 为分子黏性系数，∇_3^2 为三维拉普拉斯算子，A_z 为湍流交换系数，G 为引力常数，M 为地球的质量，$\boldsymbol{r}$ 为空气微团的位置矢量.

将式(3)—(7)代入式(2)，经整理得到旋转坐标系中矢量形式的动量方程：

$$\frac{\mathrm{d}\boldsymbol{V}_3}{\mathrm{d}t}=-\frac{1}{\rho}\nabla_3 p-2\boldsymbol{\Omega}\times\boldsymbol{V}_3+\boldsymbol{g}+\boldsymbol{F}, \tag{8}$$

其中，$\boldsymbol{F}=\boldsymbol{F}_2+\boldsymbol{F}_3$ 为分子黏性力与湍流黏性力之和，即摩擦力.用 $\boldsymbol{V}_3$ 点乘式(8)两端得到旋转坐标系中的动能方程：

$$\frac{\mathrm{d}K}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{V}_3^2}{2}\right)=-\frac{1}{\rho}\boldsymbol{V}_3\cdot\nabla_3 p-\boldsymbol{g}w+\boldsymbol{F}\cdot\boldsymbol{V}_3, \tag{9}$$

其中，$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\boldsymbol{V}_3\cdot\nabla_3$，$K=\frac{1}{2}(u^2+v^2+w^2)$ 为动能，式(9)左边是动能的变化率，右边三项分别是水平气压梯度力、重力所作的功率和摩擦消耗的功率.

定义能量泛函 $I(u,v,w,p)\in C^2(\Omega)$，

$$I(u,v,w,p)=\iiint_\Omega \frac{\partial K}{\partial t}\mathrm{d}\sigma, \tag{10}$$

其中，Ω 是任意台风区域，用该泛函表示台风强度变化率.假设在 $\partial\Omega$ 上 u,v,w,p 以及 u,v,w 关于 x,y,z 的所有一阶偏导数均为常数，此时台风在区域边界上可视为基本气流[22].对 I 作变分，有

$$\begin{aligned}
\delta I=&\iiint_\Omega \delta\frac{\partial K}{\partial t}\mathrm{d}\sigma=\iiint_\Omega \delta\left(\frac{\mathrm{d}K}{\mathrm{d}t}-\boldsymbol{V}_3\cdot\nabla_3 K\right)\mathrm{d}\sigma=\\
&\iiint_\Omega \delta\left\{-\frac{1}{\rho}\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)-gw+\frac{\mu}{\rho}(u\nabla^2 u+v\nabla^2 v+w\nabla^2 w)-\right.\\
&\left.u\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}\right)-v\left(u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial y}\right)-w\left(u\frac{\partial u}{\partial z}+v\frac{\partial v}{\partial z}+w\frac{\partial w}{\partial z}\right)\right\}\mathrm{d}\sigma=\\
&\iiint_\Omega \left\{\left[\frac{2\mu}{\rho}\nabla^2 u-\frac{1}{\rho}\frac{\partial p}{\partial x}-\left(v\frac{\partial v}{\partial x}-u\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial x}-u\frac{\partial w}{\partial z}\right)\right]\delta u+\right.\\
&\left[\frac{2\mu}{\rho}\nabla^2 v-\frac{1}{\rho}\frac{\partial p}{\partial y}-\left(u\frac{\partial u}{\partial y}-v\frac{\partial u}{\partial x}+w\frac{\partial w}{\partial y}-v\frac{\partial w}{\partial z}\right)\right]\delta v+\\
&\left.\left[\frac{2\mu}{\rho}\nabla^2 w-\frac{1}{\rho}\frac{\partial p}{\partial z}-g-\left(u\frac{\partial u}{\partial z}-w\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial z}-w\frac{\partial v}{\partial y}\right)\right]\delta w\right\}\mathrm{d}\sigma=\\
&\iiint_\Omega \left[\left(\frac{2\mu}{\rho}\nabla_3^2 u-\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{\partial K}{\partial x}\right)\delta u+\left(\frac{2\mu}{\rho}\nabla_3^2 v-\frac{1}{\rho}\frac{\partial p}{\partial y}-\frac{\partial K}{\partial y}\right)\delta v+\right.\\
&\left.\left(\frac{2\mu}{\rho}\nabla_3^2 w-\frac{1}{\rho}\frac{\partial p}{\partial z}-g-\frac{\partial K}{\partial z}\right)\delta w\right]\mathrm{d}\sigma.
\end{aligned} \tag{11}$$

令 $\delta I = 0$, 利用 $\delta u, \delta v, \delta w$ 的任意性得到 Euler-Lagrange 方程为

$$\frac{2\mu}{\rho}\nabla_3^2 u - \frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial K}{\partial x} = 0, \tag{12}$$

$$\frac{2\mu}{\rho}\nabla_3^2 v - \frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\partial K}{\partial y} = 0, \tag{13}$$

$$\frac{2\mu}{\rho}\nabla_3^2 w - \frac{1}{\rho}\frac{\partial p}{\partial z} - g - \frac{\partial K}{\partial z} = 0. \tag{14}$$

矢量形式为

$$2\boldsymbol{F} + \boldsymbol{f}_p - \boldsymbol{g} - \nabla_3 K = 0, \tag{15}$$

其中,$\boldsymbol{f}_p$ 为气压梯度力,$\boldsymbol{g} = (0,0,g)$.式(15)表明,当台风强度变化率达到最大时,摩擦力、气压梯度力、重力和动能梯度满足四力平衡.因此,通过这四个力确定的向量可以作为台风强度的预报因子,更准确地确定台风系统强度变化率达到最大的时间点,研究台风强度的变化规律.此外,将式(15)变形成

$$\boldsymbol{F} = \frac{1}{2}(\boldsymbol{g} + \nabla_3 K - \boldsymbol{f}_p), \tag{16}$$

则台风强度变化最快时的摩擦力表达式由式(16)给出,它为计算较困难且常常被忽略的摩擦力提供了一种新的计算方法.

2 台风风场分解

2.1 变分提取有旋气流

为了进一步研究台风强度变化最快时涡度的形式和流场的变化,采用黄思训等[15]的方法对台风风场进行分解.首先将风场 $u(x,y,z)$,$v(x,y,z)$ 分解成平面场 $u'(x,y)$,$v'(x,y)$ 和垂直场 $H(z)$,即

$$u(x,y,z) = u'(x,y)H(z), \tag{17}$$

$$v(x,y,z) = v'(x,y)H(z), \tag{18}$$

其中,风场 u, v 对应台风强度变化率最大的时刻,分别满足方程(12)和(13).由于台风的涡旋运动受环境气流引导,因此台风流场可以分解成有旋流场和无旋流场.利用变分方法从风场 u', v' 中提取无旋气流 $u_1(x,y)$,$v_1(x,y)$[15],使得泛函

$$J_1(u',v',u_1,v_1) = \frac{1}{2}\iiint_\Omega [(u'-u_1)^2 + (v'-v_1)^2]\,\mathrm{d}\Omega = \min! \tag{19}$$

且 u', v', u_1, v_1 满足

$$\frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y} = 0, \tag{20}$$

$$\frac{2\mu}{\rho}\nabla_3^2(u'H) - \frac{1}{\rho}\frac{\partial p}{\partial x} - H^2\left(u'\frac{\partial u'}{\partial x} + v'\frac{\partial v'}{\partial x}\right) - w\frac{\partial w}{\partial x} = 0, \tag{21}$$

$$\frac{2\mu}{\rho}\nabla_3^2(v'H) - \frac{1}{\rho}\frac{\partial p}{\partial y} - H^2\left(u'\frac{\partial u'}{\partial y} + v'\frac{\partial v'}{\partial y}\right) - w\frac{\partial w}{\partial y} = 0. \tag{22}$$

这是一个条件变分问题,引入 Lagrange 乘子 $\lambda_1(x,y,z)$,$\lambda_2(x,y,z)$,$\lambda_3(x,y,z)$,有

$$\begin{aligned} J_1 = \iiint_\Omega \{ & \frac{1}{2}[(u'-u_1)^2 + (v'-v_1)^2] - \\ & \lambda_1\left(\frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y}\right) - \lambda_2\left[\frac{2\mu}{\rho}\nabla_3^2(u'H) - \right. \\ & \left.\frac{1}{\rho}\frac{\partial p}{\partial x} - H^2\left(u'\frac{\partial u'}{\partial x} + v'\frac{\partial v'}{\partial x}\right) - w\frac{\partial w}{\partial x}\right] - \\ & \lambda_3\left[\frac{2\mu}{\rho}\nabla_3^2(v'H) - \frac{1}{\rho}\frac{\partial p}{\partial y} - H^2\left(u'\frac{\partial u'}{\partial y} + v'\frac{\partial v'}{\partial y}\right) - \right. \\ & \left. w\frac{\partial w}{\partial y}\right]\}\,\mathrm{d}\Omega = \min! \end{aligned} \tag{23}$$

于是

$$\begin{aligned} \delta J_1 = \delta\iiint_\Omega \Bigg\{ & \frac{1}{2}[(u'-u_1)^2 + (v'-v_1)^2] - \\ & \lambda_1\left(\frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y}\right) - \lambda_2\left[\frac{2\mu}{\rho}\nabla_3^2(u'H) - \frac{1}{\rho}\frac{\partial p}{\partial x} - \right. \\ & \left. H^2\left(u'\frac{\partial u'}{\partial x} + v'\frac{\partial v'}{\partial x}\right) - w\frac{\partial w}{\partial x}\right] - \\ & \lambda_3\left[\frac{2\mu}{\rho}\nabla_3^2(v'H) - \frac{1}{\rho}\frac{\partial p}{\partial y} - \right. \\ & \left. H^2\left(u'\frac{\partial u'}{\partial y} + v'\frac{\partial v'}{\partial y}\right) - w\frac{\partial w}{\partial y}\right]\Bigg\}\,\mathrm{d}\Omega = 0. \end{aligned} \tag{24}$$

假设在 $\partial\Omega$ 上 $\lambda_1, \lambda_2, \lambda_3$ 以及 λ_2, λ_3 关于 x, y, z 的所有一阶偏导数均为 0,则有

$$\begin{aligned} \delta J_1 = \iiint_\Omega \Bigg\{ & \left[u' - u_1 - \frac{2\mu}{\rho}H\left(\frac{\partial^2\lambda_2}{\partial x^2} + \frac{\partial^2\lambda_2}{\partial y^2}\right) - \frac{2\mu}{\rho}\lambda_2\frac{\mathrm{d}^2 H}{\mathrm{d}z^2} - u'H^2\left(\frac{\partial\lambda_2}{\partial x} + \frac{\partial\lambda_3}{\partial y}\right)\right]\delta u + \\ & \left[v' - v_1 - \frac{2\mu}{\rho}H\left(\frac{\partial^2\lambda_3}{\partial x^2} + \frac{\partial^2\lambda_3}{\partial y^2}\right) - \frac{2\mu}{\rho}\lambda_3\frac{\mathrm{d}^2 H}{\mathrm{d}z^2} - v'H^2\left(\frac{\partial\lambda_2}{\partial x} + \frac{\partial\lambda_3}{\partial y}\right)\right]\delta v - \\ & w\left(\frac{\partial\lambda_2}{\partial x} + \frac{\partial\lambda_3}{\partial y}\right)\delta w + \left(u_1 - u' - \frac{\partial\lambda_1}{\partial y}\right)\delta u_1 + \left(v_1 - v' + \frac{\partial\lambda_1}{\partial x}\right)\delta v_1 - \end{aligned}$$

$$\frac{1}{\rho}\left(\frac{\partial \lambda_2}{\partial x}+\frac{\partial \lambda_3}{\partial y}\right)\delta p-\left[\frac{2\mu}{\rho}\lambda_2\left(\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}\right)-2\lambda_2 H\left(u'\frac{\partial u'}{\partial x}+v'\frac{\partial v'}{\partial x}\right)+\frac{2\mu}{\rho}\lambda_3\left(\frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y^2}\right)-\right.$$
$$\left.2\lambda_3 H\left(u'\frac{\partial u'}{\partial y}+v'\frac{\partial v'}{\partial y}\right)+\frac{2\mu}{\rho}\left(u'\frac{\partial^2\lambda_2}{\partial z^2}+v'\frac{\partial^2\lambda_3}{\partial z^2}\right)\right]\delta H-\left(\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}\right)\delta\lambda_1-$$
$$\left[\frac{2\mu}{\rho}\nabla_3^2(u'H)-\frac{1}{\rho}\frac{\partial p}{\partial x}-H^2\left(u'\frac{\partial u'}{\partial x}+v'\frac{\partial v'}{\partial x}\right)-w\frac{\partial w}{\partial x}\right]\delta\lambda_2-$$
$$\left.\left[\frac{2\mu}{\rho}\nabla_3^2(v'H)-\frac{1}{\rho}\frac{\partial p}{\partial y}-H^2\left(u'\frac{\partial u'}{\partial y}+v'\frac{\partial v'}{\partial y}\right)-w\frac{\partial w}{\partial y}\right]\delta\lambda_3\right\}\mathrm{d}\Omega=0. \tag{25}$$

利用$\delta u,\delta v,\delta w,\delta u_1,\delta v$的任意性,可得它们对应的 Euler-Lagrange 方程分别为

$$u'-u_1-\frac{2\mu}{\rho}H\left(\frac{\partial^2\lambda_2}{\partial x^2}+\frac{\partial^2\lambda_2}{\partial y^2}\right)-\frac{2\mu}{\rho}\lambda_2\frac{\mathrm{d}^2H}{\mathrm{d}z^2}-u'H^2\left(\frac{\partial\lambda_2}{\partial x}+\frac{\partial\lambda_3}{\partial y}\right)=0, \tag{26}$$

$$v'-v_1-\frac{2\mu}{\rho}H\left(\frac{\partial^2\lambda_3}{\partial x^2}+\frac{\partial^2\lambda_3}{\partial y^2}\right)-\frac{2\mu}{\rho}\lambda_3\frac{\mathrm{d}^2H}{\mathrm{d}z^2}-v'H^2\left(\frac{\partial\lambda_2}{\partial x}+\frac{\partial\lambda_3}{\partial y}\right)=0, \tag{27}$$

$$w\left(\frac{\partial\lambda_2}{\partial x}+\frac{\partial\lambda_3}{\partial y}\right)=0, \tag{28}$$

$$u_1-u'-\frac{\partial\lambda_1}{\partial y}=0, \tag{29}$$

$$v_1-v'+\frac{\partial\lambda_1}{\partial x}=0, \tag{30}$$

$$\frac{1}{\rho}\left(\frac{\partial\lambda_2}{\partial x}+\frac{\partial\lambda_3}{\partial y}\right)=0, \tag{31}$$

$$\frac{2\mu}{\rho}\lambda_2\left(\frac{\partial^2u'}{\partial x^2}+\frac{\partial^2u'}{\partial y^2}\right)-2\lambda_2H\left(u'\frac{\partial u'}{\partial x}+v'\frac{\partial v'}{\partial x}\right)+\frac{2\mu}{\rho}\lambda_3\left(\frac{\partial^2v'}{\partial x^2}+\frac{\partial^2v'}{\partial y^2}\right)-2\lambda_3H\left(u'\frac{\partial u'}{\partial y}+v'\frac{\partial v'}{\partial y}\right)+\frac{2\mu}{\rho}\left(u'\frac{\partial^2\lambda_2}{\partial z^2}+v'\frac{\partial^2\lambda_3}{\partial z^2}\right)=0, \tag{32}$$

$$\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}=0, \tag{33}$$

$$\frac{2\mu}{\rho}\nabla_3^2(u'H)-\frac{1}{\rho}\frac{\partial p}{\partial x}-H^2\left(u'\frac{\partial u'}{\partial x}+v'\frac{\partial v'}{\partial x}\right)-w\frac{\partial w}{\partial x}=0, \tag{34}$$

$$\frac{2\mu}{\rho}\nabla_3^2(v'H)-\frac{1}{\rho}\frac{\partial p}{\partial y}-H^2\left(u'\frac{\partial u'}{\partial y}+v'\frac{\partial v'}{\partial y}\right)-w\frac{\partial w}{\partial y}=0. \tag{35}$$

根据式(29)和(30),有旋流场(u_2,v_2)和无旋流场(u_1,v_1)的表达形式分别为

$$u_1=u'+\frac{\partial\lambda_1}{\partial y}, \tag{36}$$

$$v_1=v'-\frac{\partial\lambda_1}{\partial x}, \tag{37}$$

$$u_2=-\frac{\partial\lambda_1}{\partial y}, \tag{38}$$

$$v_2=\frac{\partial\lambda_1}{\partial x}. \tag{39}$$

由式(36)—(39)可知,无旋流场(u_1,v_1)满足$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}=\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}$,是有辐散的;而有旋流场$(u_2,v_2)$满足$\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}=0$,是无辐散的.

在有旋场中引入流函数$\psi(x,y)$,满足

$$u_2=-\frac{\partial\psi}{\partial y},v_2=\frac{\partial\psi}{\partial x}, \tag{40}$$

涡度为

$$\zeta=\frac{\partial v_2}{\partial x}-\frac{\partial u_2}{\partial y}=\nabla^2\psi, \tag{41}$$

其中,∇^2为二维拉普拉斯算子.

将式(30)对x作微商,式(29)对y作微商,相减并结合式(33)得

$$\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}=\nabla^2\lambda_1. \tag{42}$$

将式(35)对x作微商,式(34)对y作微商,相减并结合式(42)得

$$H\nabla^2(\nabla^2\lambda_1)+\nabla^2\lambda_1\frac{\mathrm{d}^2H}{\mathrm{d}z^2}=0. \tag{43}$$

将式(33)、(41)和(42)联立得

$$\psi=\lambda_1. \tag{44}$$

因此,在平面风场中,二维 Lagrange 乘子$\lambda_1(x,y)$即为有旋流场的流函数$\psi(x,y)$.故方程(43)可改写为

$$H\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)+\frac{\mathrm{d}^2H}{\mathrm{d}z^2}\zeta=0. \tag{45}$$

由于$\zeta=\zeta(x,y)$,$H=H(z)$,对方程(45)采用分离变量法,并设:

$$\frac{\mathrm{d}^2 H}{\mathrm{d}z^2} = \lambda H, \quad 0 < z < z_1, \tag{46}$$

则:

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} = -\lambda \zeta, \tag{47}$$

其中,λ 为常数.

由式(31)得

$$\frac{\partial \lambda_2}{\partial x} + \frac{\partial \lambda_3}{\partial y} = 0. \tag{48}$$

将式(46)、(48)代入式(26)、(27)得

$$u' - u_1 = \frac{2\mu}{\rho} H(\nabla^2 \lambda_2 + \lambda \lambda_2), \tag{49}$$

$$v' - v_1 = \frac{2\mu}{\rho} H(\nabla^2 \lambda_3 + \lambda \lambda_3). \tag{50}$$

给定方程(46)的边界条件为 $H(0) = H$,$H(z_1) = H_1$,方程(46) 的解为

$$H(z) = \begin{cases} \dfrac{H_1 - H_0 \mathrm{e}^{-\sqrt{\lambda} z_1}}{2\mathrm{sh}(\sqrt{\lambda} z_1)} \mathrm{e}^{\sqrt{\lambda} z} + \dfrac{H_0 \mathrm{e}^{\sqrt{\lambda} z_1} - H_1}{2\mathrm{sh}(\sqrt{\lambda} z_1)} \mathrm{e}^{-\sqrt{\lambda} z}, & \lambda > 0, \\ \dfrac{H_1 - H_0}{z_1} z + H_0, & \lambda = 0, \\ \dfrac{H_1 - H_0 \cos\sqrt{-\lambda} z_1}{\sin\sqrt{-\lambda} z_1} \sin\sqrt{-\lambda} z + \\ \quad H_0 \cos\sqrt{-\lambda} z, & \lambda < 0, \end{cases} \tag{51}$$

其中,双曲正弦函数 $\mathrm{sh}\, x = \dfrac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$.考虑到在实际台风构成中,$H(z)$ 代表风廓线,表征台风的垂直结构,一般随高度的增加而增大[23],故不讨论 $\lambda < 0$ 的情况.由于

$$\lim_{\lambda \to 0} \left[\frac{H_1 - H_0 \mathrm{e}^{-\sqrt{\lambda} z_1}}{2\mathrm{sh}(\sqrt{\lambda} z_1)} \mathrm{e}^{\sqrt{\lambda} z} + \frac{H_0 \mathrm{e}^{\sqrt{\lambda} z_1} - H_1}{2\mathrm{sh}(\sqrt{\lambda} z_1)} \mathrm{e}^{-\sqrt{\lambda} z} \right] =$$

$$\lim_{\lambda \to 0} \frac{2zH_1 \mathrm{ch}(\sqrt{\lambda} z) + 2H_0(z_1 - z)\mathrm{ch}[\sqrt{\lambda}(z_1 - z)]}{2z_1 \mathrm{ch}(\sqrt{\lambda} z_1)} =$$

$$\frac{H_1 - H_0}{z_1} z + H_0, \tag{52}$$

其中,双曲余弦函数 $\mathrm{ch}\, x = \dfrac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$.因此,$\lambda > 0$ 时 $H(z)$ 的表达式在 $\lambda \to 0$ 时的值即为 $\lambda = 0$ 时 $H(z)$ 的值.故 $\lambda = 0$ 时 $H(z)$ 也符合垂直廓线的分布规律.以下讨论基于 $\lambda \geqslant 0$.

2.2 涡度的解析解

根据黄思训等[15],台风流场一般在圆域中经变分分解能提取到最大涡旋,故将方程(47)转化成极坐标 (r,θ) 下的方程

$$\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \zeta}{\partial \theta^2} = -\lambda \zeta,$$

$$0 < r < r_0, 0 \leqslant \theta \leqslant 2\pi. \tag{53}$$

为满足齐次边界条件,作变换

$$\eta(r,\theta) = \zeta(r,\theta) - \frac{r}{r_0}\zeta(r_0,\theta), \tag{54}$$

则 $\eta(0,\theta) = \eta(r_0,\theta) = 0$.记 $\zeta(r_0,\theta) = \zeta_0$,方程(53)转化成如下形式:

$$\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial \theta^2} + \frac{1}{r_0}\left(\frac{1}{r} + \lambda r\right)\zeta_0 +$$

$$\frac{1}{r_0 r}\frac{\partial^2 \zeta_0}{\partial \theta^2} = -\lambda \eta, 0 < r < r_0, 0 \leqslant \theta \leqslant 2\pi. \tag{55}$$

令 $\eta(r,\theta) = R(r)\Phi(\theta)$,$\zeta(r,\theta) = \rho(r)\Phi(\theta)$,则有

$$\rho(r) - \frac{r}{r_0}\rho(r_0) = R(r), \tag{56}$$

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2}\Phi + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r}\Phi + \frac{1}{r^2}R\frac{\mathrm{d}^2 \Phi}{\mathrm{d}\theta^2} +$$

$$\frac{\rho(r_0)}{r_0}\left(\frac{1}{r} + \lambda r\right)\Phi \frac{\rho(r_0)}{r_0 r}\frac{\mathrm{d}^2 \Phi}{\mathrm{d}\theta^2} = -\lambda R\Phi. \tag{57}$$

对方程(57)作变量分离得

$$\frac{-\dfrac{\mathrm{d}^2 \Phi}{\mathrm{d}\theta^2}}{\Phi} = \frac{\dfrac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}R}{\mathrm{d}r} + \lambda R + \dfrac{\rho(r_0)}{r_0}\left(\dfrac{1}{r} + \lambda r\right)}{\dfrac{1}{r^2}R + \dfrac{\rho(r_0)}{r_0 r}} = \mu, \tag{58}$$

其中,μ 为常数.结合 $\eta(r,\theta)$ 关于 θ 的周期性,可得

$$\begin{cases} \dfrac{\mathrm{d}^2 \Phi}{\mathrm{d}\theta^2} + \mu \Phi = 0, 0 \leqslant \theta \leqslant 2\pi, \\ \Phi(\theta) = \Phi(\theta + 2\pi), \end{cases} \tag{59}$$

$$\begin{cases} \dfrac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}R}{\mathrm{d}r} + \left(\lambda - \dfrac{\mu}{r^2}\right)R + \\ \quad \dfrac{\rho(r_0)}{r_0}\left[\lambda r - (\mu - 1)\dfrac{1}{r}\right] = 0, \quad 0 < r < r_0, \\ R(0) = 0, R(r_0) = 0. \end{cases} \tag{60}$$

式(59)是带有周期性条件的特征值问题,其特征值和特征函数分别为

$$\mu_n = n^2, \quad \Phi_n(\theta) = \{\cos n\theta, \sin n\theta\}. \tag{61}$$

其中,$n \geqslant 0$.它的解为

$$\Phi = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta, \tag{62}$$

其中,a_n,b_n 为常数.

记 $h(r)=\frac{\rho(r_0)}{r_0}\left[(n^2-1)\frac{1}{r}-\lambda r\right]$,将 $\mu_n=n^2$ 代入方程(60)中得

$$\begin{cases}\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r}+\left(\lambda-\frac{n^2}{r^2}\right)R=h(r), \quad 0<r<r_0,\\ R(0)=0, R(r_0)=0.\end{cases} \tag{63}$$

当 $\lambda=0$ 时,方程(63)为非齐次欧拉方程,它在边界条件下的解为

$$R(r)=\frac{\rho(r_0)}{r_0^n}r^n-\frac{\rho(r_0)}{r_0}r. \tag{64}$$

此时,结合式(56)得涡度 ζ 的表达式为

$$\zeta=\sum_{n=1}^{\infty}\left[(A'_n\cos n\theta+B'_n\sin n\theta)\frac{\rho(r_0)}{r_0^n}r^n\right]+c_0, \tag{65}$$

其中,A'_n, B'_n, c_0 为常数.

当 $\lambda>0$ 时,采用常数变易法[24]得方程(63)的通解为

$$R(r)=c_1J_n(\sqrt{\lambda}r)+c_2N_n(\sqrt{\lambda}r)+R^*(r), \tag{66}$$

其中,c_1, c_2 为常数,$J_n(x)=\left(\frac{x}{2}\right)^n\sum_{i=0}^{\infty}(-1)^i\cdot\frac{1}{i!\ \Gamma(i+n+1)}\left(\frac{x}{2}\right)^{2i}$ 为第一类贝塞尔函数,$N_n(x)=\frac{J_n(x)\cos(n\pi)-J_{-n}(x)}{\sin(n\pi)}$ 为第二类贝塞尔函数,$R^*(r)$ 满足

$$R^*(r)=-\frac{\pi}{2}\sqrt{\lambda}\left[J_n(\sqrt{\lambda}r)\int_0^{r_0}h(r)N_n(\sqrt{\lambda}r)r\mathrm{d}r+N_n(\sqrt{\lambda}r)\int_0^{r_0}h(r)J_n(\sqrt{\lambda}r)r\mathrm{d}r\right]. \tag{67}$$

由边界条件得

$$R(r)=\left(c_1-\frac{\delta_m^{(n)}\pi}{2r_0}\int_0^{r_0}h(r)N_n\left(\frac{\delta_m^{(n)}}{r_0}r\right)r\mathrm{d}r\right)J_n\left(\frac{\delta_m^{(n)}}{r_0}r\right), \tag{68}$$

其中,$\delta_m^{(n)}$ 为 $J_n(x)$ 的第 m 个正零点[25].方便起见,记 $A=c_1-\frac{\delta_m^{(n)}\pi}{2r_0}\int_0^{r_0}h(r)N_n\left(\frac{\delta_m^{(n)}}{r_0}r\right)r\mathrm{d}r$.此时,涡度 ζ 的表达式为

$$\zeta=\sum_{n=0}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\left[AJ_n\left(\frac{\delta_m^{(n)}}{r_0}r\right)+\frac{\rho(r_0)}{r_0}r\right]. \tag{69}$$

因此,台风强度变化率最大时有旋流场涡度的解析解为

$$\zeta=\begin{cases}\sum_{n=1}^{\infty}\left[(A'_n\cos n\theta+B'_n\sin n\theta)\frac{\rho(r_0)}{r_0^n}r^n\right]+c_0,\\ \qquad\lambda=0,\\ \sum_{n=0}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\left[AJ_n\left(\frac{\delta_m^{(n)}}{r_0}r\right)+\right.\\ \qquad\left.\frac{\rho(r_0)}{r_0}r\right], \quad\lambda>0.\end{cases} \tag{70}$$

涡度拟能的表达式为

$$\frac{1}{2}\zeta^2=\begin{cases}\frac{1}{2}\left\{\sum_{n=1}^{\infty}\left[(A'_n\cos n\theta+B'_n\sin n\theta)\frac{\rho(r_0)}{r_0^n}r^n\right]+c_0\right\}^2,\\ \qquad\lambda=0,\\ \frac{1}{2}\left\{\sum_{n=0}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\left[AJ_n\left(\frac{\delta_m^{(n)}}{r_0}r\right)+\right.\right.\\ \qquad\left.\left.\frac{\rho(r_0)}{r_0}r\right]\right\}^2, \quad\lambda>0.\end{cases} \tag{71}$$

2.3 有旋流场的解析解

当 $\lambda=0$ 时,流函数 $\psi(r,\theta)$ 满足:

$$\begin{cases}\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=\\ \qquad\sum_{n=1}^{\infty}\left[(A'_n\cos n\theta+B'_n\sin n\theta)\frac{\rho(r_0)}{r_0^n}r^n\right]+c_0,\\ \psi(r_0,\theta)=0.\end{cases} \tag{72}$$

根据 $\zeta(r,\theta)$ 的表达式,$\psi(r,\theta)$ 具有如下形式:

$$\psi(r,\theta)=\sum_{n=1}^{\infty}[C(r)\cos n\theta+D(r)\sin n\theta]+E(r)\sin^2n\theta+F(r)\cos^2n\theta+G(r)\sin n\theta\cos n\theta. \tag{73}$$

由 $\lambda_1|_{\partial\Omega}=0$ 可知 $\psi|_{\partial\Omega}=0$,故方程(72)的边界条件为

$$C(r_0)=D(r_0)=E(r_0)=F(r_0)=G(r_0)=0. \tag{74}$$

将式(73)代入方程(72),整理得

$$\frac{\mathrm{d}^2C}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}C}{\mathrm{d}r}-\frac{n^2}{r^2}C=A'_n\frac{\rho(r_0)}{r_0^n}r^n, \tag{75}$$

$$\frac{\mathrm{d}^2D}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}D}{\mathrm{d}r}-\frac{n^2}{r^2}D=B'_n\frac{\rho(r_0)}{r_0^n}r^n, \tag{76}$$

$$\frac{\mathrm{d}^2E}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}E}{\mathrm{d}r}-\frac{2n^2}{r^2}(E-F)=c_0, \tag{77}$$

$$\frac{\mathrm{d}^2F}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}F}{\mathrm{d}r}+\frac{2n^2}{r^2}(E-F)=c_0, \tag{78}$$

$$\frac{\mathrm{d}^2G}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}G}{\mathrm{d}r}-\frac{4n^2}{r^2}G=0. \tag{79}$$

方程(75)、(76)和(79)满足边界条件的解分

别为

$$C(r)=\frac{A_n\rho(r_0)}{n+1}\left(\frac{r}{r_0}\right)^n(r^2-r_0^2),\tag{80}$$

$$D(r)=\frac{B_n\rho(r_0)}{n+1}\left(\frac{r}{r_0}\right)^n(r^2-r_0^2),\tag{81}$$

$$G(r)=0,\tag{82}$$

其中 $A_n=\frac{1}{4}A'_n$,$B_n=\frac{1}{4}B'_n$.令 $E(r)+F(r)=H(r)$,$E(r)-F(r)=J(r)$,将式(77)与式(78)分别相加相减得

$$\frac{\mathrm{d}^2H}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}H}{\mathrm{d}r}=2c_0,\tag{83}$$

$$\frac{\mathrm{d}^2J}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}J}{\mathrm{d}r}-\frac{4k^2}{r^2}J=0,\tag{84}$$

且根据式(74),边界条件为 $H(r_0)=J(r_0)=0$.解得

$$H(r)=\frac{c_0}{2}(r^2-r_0^2),\tag{85}$$

$$J(r)=0.\tag{86}$$

由式(85)和(86)得

$$E(r)=F(r)=\frac{c_0}{4}(r^2-r_0^2).\tag{87}$$

因此,流函数 $\psi(r,\theta)$ 的表达式为

$$\psi(r,\theta)=\sum_{n=1}^{\infty}\left[\frac{\rho(r_0)}{n+1}\left(\frac{r}{r_0}\right)^n(r^2-r_0^2)(A_n\cos n\theta+B_n\sin n\theta)\right]+\frac{c_0}{4}(r^2-r_0^2).\tag{88}$$

根据式(40),有旋场 $u_2(r,\theta)$,$v_2(r,\theta)$ 满足:

$$u_2=-\sin\theta\frac{\partial\psi}{\partial r}-\frac{\cos\theta}{r}\frac{\partial\psi}{\partial\theta},\tag{89}$$

$$v_2=\cos\theta\frac{\partial\psi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}.\tag{90}$$

因此,有旋流场 (u_2,v_2) 的表达式为

$$u_2=\sum_{n=1}^{\infty}\left\{\frac{\rho(r_0)}{r_0^n}r^{n-1}\left(r^2-\frac{nr_0{}^2}{n+1}\right)[A_n\sin(n-1)\theta-B_n\cos(n-1)\theta]-\frac{\rho(r_0)}{(n+1)r_0{}^n}r^{n+1}[A_n\sin(n+1)\theta-B_n\cos(n+1)\theta]\right\}-\frac{c_0}{2}r\sin\theta,\tag{91}$$

$$v_2=\sum_{n=1}^{\infty}\left\{\frac{\rho(r_0)}{r_0^n}r^{n-1}\left(r^2-\frac{nr_0{}^2}{n+1}\right)[A_n\cos(n-1)\theta+B_n\sin(n-1)\theta]+\frac{\rho(r_0)}{(n+1)r_0{}^n}r^{n+1}[A_n\cos(n+1)\theta+B_n\sin(n+1)\theta]\right\}+\frac{c_0}{2}r\cos\theta.\tag{92}$$

由变分的性质可知,分解得到的有旋气流 (u_2,v_2) 是最大的涡旋.

当 $\lambda>0$ 时,为方便求解流函数,对涡度 ζ 作如下简化:

$$\zeta=\sum_{n=0}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\left[\frac{(-1)^nA}{2n!\ n!}\left(\frac{\delta_m^{(n)}}{2}\right)^{3n}\left(\frac{r}{r_0}\right)^{3n}+\frac{\rho(r_0)}{r_0}r\right].\tag{93}$$

记 $E_n=\frac{(-1)^nA}{2n!\ n!}\left(\frac{\delta_m^{(n)}}{2}\right)^{3n}$,则流函数 $\psi(r,\theta)$ 满足:

$$\begin{cases}\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=\\ \quad\sum_{n=0}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\left[E_n\left(\frac{r}{r_0}\right)^{3n}+\frac{\rho(r_0)}{r_0}r\right],\\ \psi(r_0,\theta)=0.\end{cases}\tag{94}$$

同理,$\psi(r,\theta)$ 具有如下形式:

$$\psi(r,\theta)=\sum_{n=0}^{\infty}[H(r)\cos n\theta+M(r)\sin n\theta].\tag{95}$$

边界条件为

$$H(r_0)=M(r_0)=0.\tag{96}$$

将式(95)代入方程(94),整理得

$$\frac{\mathrm{d}^2H}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}H}{\mathrm{d}r}-\frac{n^2}{r^2}H=a_n\left[E_n\left(\frac{r}{r_0}\right)^{3n}+\frac{\rho(r_0)}{r_0}r\right],\tag{97}$$

$$\frac{\mathrm{d}^2M}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}M}{\mathrm{d}r}-\frac{n^2}{r^2}M=b_n\left[E_n\left(\frac{r}{r_0}\right)^{3n}+\frac{\rho(r_0)}{r_0}r\right].\tag{98}$$

记 $G_n=\frac{\rho(r_0)}{(9-n^2)r_0}$,$F_n=\frac{E_n}{(2n^2+3n+1)r_0^{3n}}$,方程(97)、(98)的解分别为

$$H(r)=a_nG_nr^3-a_n(G_nr_0{}^{3-n}+F_nr_0{}^{2n+2})r^n+a_nF_nr^{3n+2},\tag{99}$$

$$M(r)=b_nG_nr^3-b_n(G_nr_0{}^{3-n}+F_nr_0{}^{2n+2})r^n+b_nF_nr^{3n+2}.\tag{100}$$

因此,流函数 $\psi(r,\theta)$ 的表达式为

$$\psi(r,\theta)=\sum_{n=0}^{\infty}a_n[G_nr^3-(G_nr_0^{3-n}+F_nr_0^{2n+2})r^n+F_nr^{3n+2}]\cos n\theta+b_n[G_nr^3-(G_nr_0^{3-n}+F_nr_0^{2n+2})r^n+F_nr^{3n+2}]\sin n\theta.\tag{101}$$

根据式(89)、(90),有旋流场 (u_2,v_2) 的表达

式为

$$u_2 = \sum_{n=0}^{\infty} b_n[-3G_n r^2 + n(G_n {r_0}^{3-n} + F_n {r_0}^{2n+2})r^{n-1} - (3n+2)F_n r^{3n+1}]\sin\theta\sin n\theta + a_n[-3G_n r^2 + n(G_n {r_0}^{3-n} + F_n {r_0}^{2n+2})r^{n-1} - (3n+2)F_n r^{3n+1}]\sin\theta\cos n\theta + a_n[nG_n r^2 - n(G_n {r_0}^{3-n} + F_n {r_0}^{2n+2})r^{n-1} + nF_n r^{3n+1}]\cos\theta\sin n\theta + b_n[-nG_n r^2 + n(G_n {r_0}^{3-n} + F_n {r_0}^{2n+2})r^{n-1} - nF_n r^{3n+1}]\cos\theta\cos n\theta, \tag{102}$$

$$v_2 = \sum_{n=0}^{\infty} a_n[nG_n r^2 - n(G_n {r_0}^{3-n} + F_n {r_0}^{2n+2})r^{n-1} + nF_n r^{3n+1}]\sin\theta\sin n\theta + b_n[-nG_n r^2 + n(G_n {r_0}^{3-n} + F_n {r_0}^{2n+2})r^{n-1} - nF_n r^{3n+1}]\sin\theta\cos n\theta + b_n[3G_n r^2 - n(G_n {r_0}^{3-n} + F_n {r_0}^{2n+2})r^{n-1} + (3n+2)F_n r^{3n+1}]\cos\theta\sin n\theta + a_n[3G_n r^2 - n(G_n {r_0}^{3-n} + F_n {r_0}^{2n+2})r^{n-1} + (3n+2)F_n r^{3n+1}]\cos\theta\cos n\theta. \tag{103}$$

2.4 数值试验

由于 $\lambda \geqslant 0$ 时,有旋流场 (u_2, v_2) 都是关于 $\frac{r}{r_0}$ 的级数,显然收敛,因此可取前几项作为级数和的近似.

当 $\lambda = 0$ 时,根据有旋流场的表达式(91)、(92),取 $n=1, r_0=1, A_1=1, B_1=1, c_0=16, \rho(1)=2$,台风强度变化最快时的流线图如图 1 所示.可以看出,流线图在经向和纬向上呈均匀梯度分布,符合实际流场分布[15].因此,在有旋流场的表达式(91)、(92)中,取 $n=1$ 可得有旋流场及其涡度的解析解.

当 $\lambda > 0$ 时,根据有旋流场的表达式(102)、(103),取 $n=0, m=1, r_0=1, a_0=1, b_0=1, c_1=1, \rho(1)=1$,通过贝塞尔函数的积分性质得 $\delta_1^{(0)} \approx \frac{5}{2}$, $A \approx \frac{2}{5}$.图 2 给出了 $n=0$ 时台风强度变化率最大时刻的流线图.同样地,流线图在经向和纬向上梯度分布均匀,与实际流场相吻合[15].故 $n=0$ 时有旋流场的表达式(102)、(103)是符合实测流场分布的解析解.$\lambda \geqslant 0$ 时的结果为黄思训等[15]的结论提供了有力的理论依据.

因此,台风强度变化最快时涡度、涡度拟能和流场的一个解析解分别为

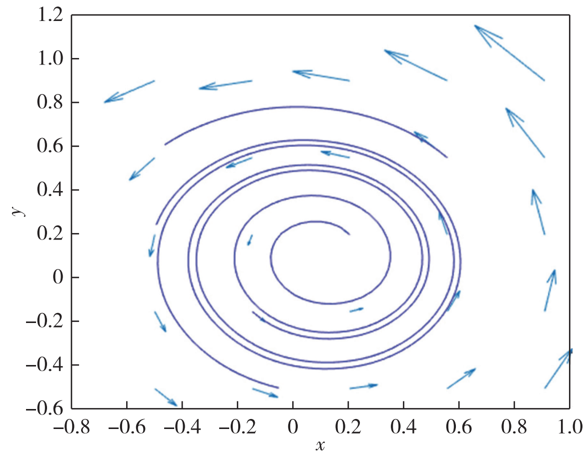

图 1　$\lambda = 0, n = 1$ 时台风强度变化率最大时刻的流线图

Fig. 1　Plot of streamline at the maximum change rate of TC intensity when $\lambda = 0, n = 1$

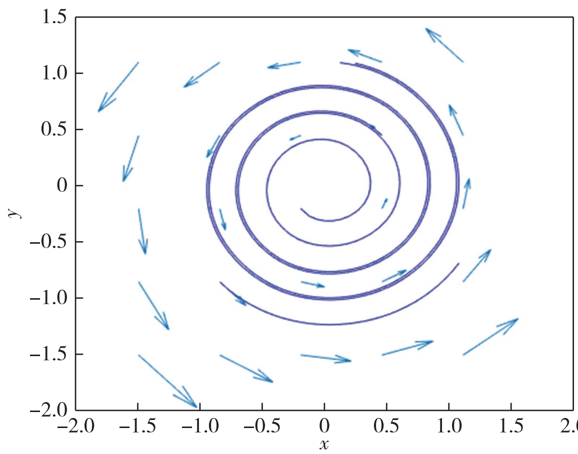

图 2　$\lambda > 0, n = 0$ 时台风强度变化率最大时刻的流线图

Fig. 2　Plot of streamline at the maximum change rate of TC intensity when $\lambda > 0, n = 0$

$$\zeta = \begin{cases} (A_1\cos\theta + B_1\sin\theta)\dfrac{\rho(r_0)}{r_0}r + c_0, \lambda = 0, \\ a_0\left[AJ_0\left(\dfrac{\delta_m^{(0)}}{r_0}r\right) + \dfrac{\rho(r_0)}{r_0}r\right], \lambda > 0, \end{cases} \tag{104}$$

$$\frac{1}{2}\zeta^2 = \begin{cases} \dfrac{1}{2}\left[(A_1\cos\theta + B_1\sin\theta)\dfrac{\rho(r_0)}{r_0}r + c_0\right]^2, \lambda = 0, \\ \dfrac{1}{2}a_0^2\left[AJ_0\left(\dfrac{\delta_m^{(0)}}{r_0}r\right) + \dfrac{\rho(r_0)}{r_0}r\right]^2, \lambda > 0, \end{cases} \tag{105}$$

$$u_2 = \begin{cases} -\dfrac{\rho(r_0)}{2r_0}r^2(A_2\sin 2\theta - B_2\cos 2\theta + 2B_1) - \dfrac{c_0}{2}r\sin\theta + \dfrac{1}{2}B_1 r_0\rho(r_0), \lambda = 0, \\ a_0(-3G_0 r^2 - 2F_0 r)\sin\theta, \lambda > 0, \end{cases} \tag{106}$$

$$v_2=\begin{cases}\dfrac{\rho(r_0)}{2r_0}r^2[A_1\cos 2\theta+B_1\sin 2\theta+2A_1]+\\ \quad\dfrac{c_0}{2}r\cos\theta-\dfrac{1}{2}A_1r_0\rho(r_0),\lambda=0,\\ a_0(3G_0r^2+2F_0r)\cos\theta,\lambda>0,\end{cases}\tag{107}$$

其中, $G_0=\dfrac{\rho(r_0)}{9r_0}$, $F_0=\dfrac{1}{r_0^{3n}}\left(c_1-\dfrac{\delta_m^{(n)}\pi}{2r_0}\int_0^{r_0}h(r)N_n\cdot\left(\dfrac{\delta_m^{(n)}}{r_0}r\right)r\mathrm{d}r\right)$.

有旋流场(u_2,v_2)更直观清晰地反映台风每一层的流场结构,涡度和涡度拟能的变化有助于研究台风的运动机理和风场中能量的变化.台风的整体结构取决于垂直廓线$H(z)$,而λ是决定$H(z)$的主要参数,因此λ决定整个台风的层次结构.

3 结论

本文从理论层面研究台风强度变化率最大时台风能量满足的关系,给出台风强度变化最快时涡度的解析解.结果表明,当台风强度变化率达到最大时,摩擦力、气压梯度力、重力和动能梯度满足四力平衡.通过这四个力确定的向量可以作为台风强度的预报因子,更准确地确定台风系统强度变化率达到最大的时间点,研究台风强度的变化规律.在四力平衡关系的基础上,进一步通过风场变分分解提取到有旋场中的最大涡旋,得到台风强度变化最快时涡度和流场的一个解析解,从理论上证明了台风发展最快时前人的研究[15]结果,对研究台风发展过程尤其是平衡过程中的运动机理和台风的空间结构具有一定的指导意义,为台风路径和强度预报提供了一定的理论指引.此外,本文的结论为计算较困难且常常被忽略的摩擦力提供了一种新的计算方法.

参考文献
References

[1] Wang Y, Wu C C. Current understanding of tropical cyclone structure and intensity changes: a review[J]. Meteorology and Atmospheric Physics, 2004, 87: 257-278

[2] 端义宏,余晖,伍荣生.热带气旋强度变化研究进展[J].气象学报,2005,63(5):636-645
DUAN Yihong, YU Hui, WU Rongsheng. Review of the research in the intensity change of typhoon cyclone[J]. Acta Meteorologica Sinica, 2005, 63(5): 636-645

[3] Hazelton A T, Zhang X J, Gopalakrishnan S, et al. High-resolution ensemble HFV3 forecasts of hurricane Michael (2018): rapid intensification in shear[J]. Monthly Weather Review, 2020, 148(5): 2009-2032

[4] Pandey R S, Liou Y A, Liu J C. Season-dependent variability and influential environmental factors of super-typhoons in the northwest Pacific basin during 2013−2017 [J]. Weather and Climate Extremes, 2021, 31 (7051): 100307

[5] Yang H, Xu G Y, Wang X F, et al. Quantitative analysis of water vapor transport during Mei-yu front rainstorm period over the Tibetan Plateau and Yangtze-Huai River Basin[J]. Advances in Meteorology, 2019, 2019(2): 1-14

[6] Yu J Q, Gao S, Zhang L, et al. Analysis of a remote rainstorm in the Yangtze River Delta region caused by typhoon Mangkhut (2018)[J]. Journal of Marine Science and Engineering, 2020, 8(5): 345

[7] Done J M, Ge M, Holland G J, et al. Modelling global tropical cyclone wind footprints[J]. Natural Hazards and Earth System Sciences, 2020, 20(2): 567-580

[8] Wada A, Kanada S, Yamada H. Effect of air-sea environmental conditions and interfacial processes on extremely intense typhoon Haiyan (2013) [J]. Journal of Geophysical Research: Atmospheres, 2018, 123, 10379-10405

[9] Prakash K R, Pant V, Nigam T. Effects of the sea surface roughness and sea spray-induced flux parameterization on the simulations of a tropical cyclone[J]. Journal of Geophysical Research: Atmospheres, 2019, 124: 14037-14058

[10] Arnold V I. Conditions for non-linear stability of stationary plane curvilinear flows of an ideal fluid[J]. Dokl. Akad. Nauk SSSR, 1965, 162: 975-978

[11] Vallis G K. Mechanisms and parameterizations of geostrophic adjustment and a variational approach to balanced flow[J]. Journal of the Atmospheric Sciences, 1992, 49: 1144-1160

[12] 伍荣生.Rossby 波的能量、能量通量与 Lagrange 函数[J].气象学报,1986,44(2):158-165
WU Rongsheng. Energy, energy flux and lagrangian of rossby wave [J]. Acta Meteorologica Sinica, 1986, 44 (2): 32-39

[13] Finlayson B A. The method of weighted residuals and variational principles [M]. Pittsburgh, PA: Academic Press, 1972

[14] Barth A, Beckers J M, Alvera-Azcarate A, et al. Filtering inertia-gravity waves from the initial conditions of the linear shallow water equations[J]. Ocean Modelling, 2007, 19: 204-218

[15] 黄思训,蔡其发,项杰,等.台风风场分解[J].物理学报,2007,56(5):3022-3027
HUANG Sixun, CAI Qifa, XIANG Jie, et al. On decomposition of typhoon flow field[J]. Acta Physica Sinica, 2007, 56(5): 3022-3027

[16] Wang S C, Huang S X. Sufficient conditions of Rayleigh-Taylor stability and instability in equatorial ionosphere [J]. Applied Mathematics and Mechanics, 2016, 37: 181-192

[17] Badin G, Crisciani F. Variational principles in geophysical fluid dynamics and approximated equations[M]//Variational Formulation of Fluid and Geophysical Fluid Dynamics. Cham, Switzerland: Springer, 2018: 135-182

[18] Mohan G M, Srinivas C V, Naidu C V, et al. Real-time

numerical simulation of tropical cyclone Nilam with WRF:experiments with different initial conditions,3D-Var and ocean mixed layer model[J].Natural Hazards,2015,77:597-624

[19] Cui Z Q,Pu Z X,Tallapragada V,et al.A preliminary impact study of CYGNSS ocean surface wind speeds on numerical simulations of hurricanes[J].Geophys Res Lett,2019,46(5):2984-2992

[20] Singh K S,Bhaskaran P K.Prediction of landfalling Bay of Bengal cyclones during 2013 using the high resolution weather research and forecasting model [J]. Meteorological Applications,2020,27(1):e1850

[21] Zhang L,Tian X J,Zhang H Q,et al.Impacts of multigrid NLS-4DVar-based Doppler radar observation assimilation on numerical simulations of landfalling typhoon Haikui (2012)[J].Advances in Atmospheric Sciences,2020,37:873-892

[22] 谢义炳,陈受钧,张一良,等.东南亚基本气流与台风发生的一些事实的统计与分析[J].气象学报,1963,21(2):206-217
XIE Yibing,CHEN Shoujun,ZHANG Yiliang,et al.A preliminarily statistic and synoptic study about the basic currents over southastern Asia and the initiation of typhoons[J].Acta Meteorologica Sinica,1963,21(2):206-217

[23] 赵坤,王明筠,朱科锋,等.登陆台风边界层风廓线特征的地基雷达观测[J].气象学报,2015,73(5):837-852
ZHAO Kun,WANG Mingjun,ZHU Kefeng,et al.An analysis of the CINRAD-98D observations for the landfalling typhoon boundary layer wind profiles and their characteristics[J].Acta Meteorologica Sinica,2015,73(5):837-852

[24] 阎凯.圆环旋转粘性液体射流稳定性及破碎研究[D].北京:北京交通大学,2014
YAN Kai.Study on stability and breakup of an annular swirling viscous liquid sheet[D].Beijing:Beijing Jiaotong University,2014

[25] Asmar N H.Partial differential equations with Fourier series and boundary value problems[M].3rd ed.New York:Dover Publications,2016

Analytical solution of vorticity when tropical cyclone intensity changes the fastest

FAN Jingwei[1] ZHOU Weican[2,3] FENG Yecheng[2] GUAN Yuanhong[1]

1 School of Mathematics and Statistics,Nanjing University of Information Science & Technology,Nanjing 210044
2 Key Laboratory of Meteorological Disaster,Ministry of Education/International Joint Laboratory on Climate and Environment Change/Collaborative Innovation Center on Forecast and Evaluation of Meteorological Disasters,Nanjing University of Information Science & Technology,Nanjing 210044
3 Suqian Zeda Vocational & Technical College,Suqian 223800

Abstract In this study,the kinetic energy equation is derived from the basic equation in the rotating coordinate system,and then the energy functional describing the change rate of Tropical Cyclone (TC) intensity is defined by the local change rate of kinetic energy.Afterwards,the Euler-Lagrange equation is obtained by taking the variation of the functional.The equation shows that when the change rate of TC intensity reaches the maximum,the friction,the pressure gradient force,the gravity and the gradient of kinetic energy are balanced.Therefore,the vector determined by these four forces in the balance equation can be used as a predictor of TC intensity,which can more accurately determine the time when the intensity change rate of TCs reaches the maximum.Furthermore,the maximum vortex in a rotational field is extracted by the variational decomposition of wind field,and the analytical solution of vorticity and flow field are obtained when TC intensity changes the fastest.The conclusion has certain practical value for studying the variation trend of velocity and the spatial structure of TC in the evolution of TC,and provides certain theoretical guidance for TC track and intensity forecast.

Key words tropical cyclone intensity;vorticity;analytical solution;variational method;wind field decomposition