DOI:10.13878/j.cnki.jnuist.2021.05.004

李燕[1] 王朝航[2] 高帅斌[3]

一类非线性中立型随机延迟微分方程的截断型 θ-EM 方法

摘要

本文考虑了一类非线性中立型随机延迟微分方程,其漂移项系数和扩散项系数均是超线性增长的,且中立项满足压缩映射条件.本文建立了这类方程的截断型 θ-EM 算法,并得到了其收敛率.最后,给出一个例子验证了理论结果.

关键词

随机延迟微分方程;中立项;截断型 θ-EM 算法;强收敛率

中图分类号 O241.8

文献标志码 A

收稿日期 2021-09-14

资助项目 国家自然科学基金(62076106)

作者简介

李燕,女,博士,副教授,研究方向为随机微分方程数值解.ly@mail.hzau.edu.cn

高帅斌(通信作者),男,博士生,研究方向为随机微分方程数值解.shuaibingao@163.com

1 华中农业大学 理学院,武汉,430070

2 中南民族大学 数学与统计学学院,武汉,430074

3 上海师范大学 数理学院,上海,200233

0 引言

随机延迟微分方程在众多领域有着重要的应用.当随机微分方程不仅依赖于过去和现在的状态,也依赖于包含延迟导数和方程本身的时候,称这类方程为中立型随机延迟微分方程,对这类方程的研究更加具有现实意义[1].随机微分方程的解析解在很多情况下是无法被计算出来的,因此可以用数值解来逼近解析解,比如 Euler-Maruyama(EM)方法.当漂移项系数和扩散项系数均超线性增长时,文献[2]给出了带有常延迟的随机微分方程的 θ-EM 方法,如果 $\theta=0$,则截断型 θ-EM 算法退化为截断型 EM 算法.文献[3]给出了截断型 EM 算法及其收敛率;文献[4]得到了随机延迟微分方程的 EM 算法的收敛性;文献[5]分析了中立型随机微分方程的局部截断型 EM 算法和 θ 算法.在文献[2]的基础上,本文给出了中立型随机延迟微分方程的截断型 θ-EM 算法及其数值解的收敛率.最后,通过一个简单的例子说明了算法的有效性.

1 相关知识

在本文中,做出如下符号规定.若 $\boldsymbol{x}\in\mathbf{R}^n$,则 $|\boldsymbol{x}|$ 表示 Euclidean 范数.若 $\boldsymbol{A}$ 是矩阵,则 $|\boldsymbol{A}|$ 表示其迹范数,即 $|\boldsymbol{A}|=\sqrt{\mathrm{trace}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})}$.若 a,b 是实数,则 $a\wedge b=\min\{a,b\}$,$a\vee b=\max\{a,b\}$,$\lfloor a\rfloor$ 表示不超过 a 的最大整数.令 $\mathbf{R}_+=[0,+\infty)$.若 H 是一个集合,则 $\mathbb{I}_H$ 表示其示性函数,这意味着,当 $\omega\in H$ 时,$\mathbb{I}_H(\omega)=1$,否则 $\mathbb{I}_H(\omega)=0$.令 C 表示一个任意可变的正常实数.设 $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant0},\mathbb{P})$ 是一个完备的概率空间,其 σ 代数流 $\{\mathcal{F}_t\}_{t\geqslant0}$ 满足一般条件,$\mathbb{E}$ 是定义在 $\mathbb{P}$ 上的期望.设 $\mathcal{C}([-\rho,0];\mathbf{R}^n)$ 是从 $[-\rho,0]$ 映射到 $\mathbf{R}^n$ 上的 ν 函数族,其范数 $\|\nu\|=\sup_{-\rho\leqslant\vartheta\leqslant0}|\nu(\vartheta)|$.设 $\mathcal{L}^p_{\mathcal{F}_0}([-\rho,0];\mathbf{R}^n)$ 是 $\mathcal{F}_0$ 可测的 $\mathcal{C}([-\rho,0];\mathbf{R}^n)$ 值的 ξ 随机变量族,且 $\mathbb{E}\|\xi\|^p<\infty$.设 $\boldsymbol{B}(t)=(B_1(t),B_2(t),\cdots,B_m(t))^{\mathrm{T}}$ 是 m 维 Brownian 运动.

在本文中,考虑非线性中立型随机延迟微分方程:

$$\begin{aligned}\mathrm{d}[\boldsymbol{x}(t)-D(\boldsymbol{x}(t-\tau))]=&f(\boldsymbol{x}(t),\boldsymbol{x}(t-\tau))\mathrm{d}t+\\&g(\boldsymbol{x}(t),\boldsymbol{x}(t-\tau))\mathrm{d}\boldsymbol{B}(t),\end{aligned}\tag{1}$$

其初值

$$x_0=\xi\in\mathcal{L}^p_{\mathcal{F}_0}([-\tau,0];\mathbf{R}^n).\quad(2)$$

其中$f:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n,g:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^{n\times m},D:\mathbf{R}^n\to\mathbf{R}^n$都是连续且 Borel 可测的函数.对初值和中立项施加以下假设:

(A1) 存在常数 $K_1>0$ 和 $\gamma\in(0,1]$,使得:

$$|\xi(t)-\xi(s)|\leqslant K_1|t-s|^\gamma,\quad\forall -\tau\leqslant s,t\leqslant 0.\quad(3)$$

(A2) $D(0)=0$ 和 $D(\cdot)$ 满足压缩映射条件,即存在一个常数 $K_2\in(0,1)$,使得:

$$|D(\boldsymbol{x})-D(\boldsymbol{y})|\leqslant K_2|\boldsymbol{x}-\boldsymbol{y}|,\quad\forall\boldsymbol{x},\boldsymbol{y}\in\mathbf{R}^n.\quad(4)$$

下面开始定义中立型随机延迟微分方程(1)的截断型 θ-EM 方法.首先,选取一个严格单调递减的函数 $\varphi:(0,1]\to(0,\infty)$,使得:

$$\lim_{\Delta\to 0}\varphi(\Delta)=\infty,K_{\varphi(\Delta)}\Delta^{\frac14}\leqslant 1,\quad\forall\Delta\in(0,1],\quad(5)$$

其中,$K_{\varphi(\Delta)}$ 是依赖于 $\varphi(\Delta)$ 的函数.例如,对于某个 $\varepsilon\in(0,1/8]$,令 $\varphi(\Delta)=\Delta^{-\varepsilon}$ 和 $K_{\varphi(\Delta)}=\varphi(\Delta)$.对于给定的步长 $\Delta\in(0,1]$,定义截断映射:

$$\pi_\Delta(\boldsymbol{x})=(|\boldsymbol{x}|\wedge\varphi(\Delta))\frac{\boldsymbol{x}}{|\boldsymbol{x}|},\quad(6)$$

其中,若 $\boldsymbol{x}=0$,设 $\boldsymbol{x}/|\boldsymbol{x}|=0$.定义截断的漂移项系数和扩散项系数:

$$f_\Delta(\boldsymbol{x},\boldsymbol{y})=f(\pi_\Delta(\boldsymbol{x}),\pi_\Delta(\boldsymbol{y})),$$
$$g_\Delta(\boldsymbol{x},\boldsymbol{y})=g(\pi_\Delta(\boldsymbol{x}),\pi_\Delta(\boldsymbol{y})).\quad(7)$$

此外,不失一般性,假设存在两个正整数 M 和 M' 使得 $\Delta=\frac{\tau}{M}=\frac{T}{M'}$.定义 $t_k=k\Delta,k=-M,-M+1,\cdots,0,\cdots,M'-1$. 现在给出(1)的离散的截断型 θ-EM 算法的定义. 当 $k=-M,\cdots,0$ 时, 令 $\boldsymbol{X}_\Delta(t_k)=\xi(t_k)$,

$$\begin{aligned}\boldsymbol{X}_\Delta(t_{k+1})=&\boldsymbol{X}_\Delta(t_k)+D(\boldsymbol{X}_\Delta(t_{k+1-M}))-\\&D(\boldsymbol{X}_\Delta(t_{k-M}))+\theta f_\Delta(\boldsymbol{X}_\Delta(t_{k+1}),\boldsymbol{X}_\Delta(t_{k+1-M}))\Delta+\\&(1-\theta)f_\Delta(\boldsymbol{X}_\Delta(t_k),\boldsymbol{X}_\Delta(t_{k-M}))\Delta+\\&g_\Delta(\boldsymbol{X}_\Delta(t_k),\boldsymbol{X}_\Delta(t_{k-M}))\Delta\boldsymbol{B}_k,\end{aligned}\quad(8)$$

其中 $k=0,1,\cdots,M'-1,\Delta\boldsymbol{B}_k=\boldsymbol{B}(t_{k+1})-\boldsymbol{B}(t_k)$.定义:

$$\bar{\boldsymbol{x}}_\Delta(t)=\sum_{k=0}^{M'-1}\boldsymbol{X}_\Delta(t_k)\mathbb{I}_{[t_k,t_{k+1})}(t).\quad(9)$$

另一种连续时间的数值格式定义为

$$\begin{aligned}\boldsymbol{x}_\Delta(t)=&\xi(0)+D(\bar{\boldsymbol{x}}_\Delta(t-\tau))-D(\xi(-\tau))+\\&\theta f_\Delta(\boldsymbol{x}_\Delta(t),\boldsymbol{x}_\Delta(t-\tau))\Delta-\theta f_\Delta(\xi(0),\xi(-\tau))\Delta+\\&\int_0^t f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))\mathrm{d}s+\\&\int_0^t g_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))\mathrm{d}B(s).\end{aligned}\quad(10)$$

可以发现 $\boldsymbol{X}_\Delta(t_k)=\bar{\boldsymbol{x}}_\Delta(t_k)=\boldsymbol{x}_\Delta(t_k)$.为了方便,记:

$$\boldsymbol{y}_\Delta(t)=\boldsymbol{x}_\Delta(t)-\theta f_\Delta(\boldsymbol{x}_\Delta(t),\boldsymbol{x}_\Delta(t-\tau))\Delta \text{ 和 } \bar{\boldsymbol{y}}_\Delta(t)=\bar{\boldsymbol{x}}_\Delta(t)-\theta f_\Delta(\bar{\boldsymbol{x}}_\Delta(t),\bar{\boldsymbol{x}}_\Delta(t-\tau))\Delta.\quad(11)$$

2 主要结论

为了得到(1)的截断型 θ-EM 算法的强收敛率,需要对漂移项系数和扩散项系数施加以下假设:

(A3)存在常数 $L_R>0$,使得对任意的 $\boldsymbol{x},\boldsymbol{y},\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}\in\mathbf{R}^n$ 且 $|\boldsymbol{x}|\vee|\boldsymbol{y}|\vee|\bar{\boldsymbol{x}}|\vee|\bar{\boldsymbol{y}}|\leqslant R$,有:

$$|f(\boldsymbol{x},\boldsymbol{y})-f(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})|\vee|g(\boldsymbol{x},\boldsymbol{y})-g(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})|\leqslant L_R(|\boldsymbol{x}-\bar{\boldsymbol{x}}|+|\boldsymbol{y}-\bar{\boldsymbol{y}}|).\quad(12)$$

在(A4)之前,引入函数 $V_i,i=1,2,3$.假设存在 $K_{V_i}>0$ 和 $\beta_i\geqslant 1$,使得对任意的 $\boldsymbol{x},\boldsymbol{y}\in\mathbf{R}^n$,有:

$$0\leqslant V_i(\boldsymbol{x},\boldsymbol{y})\leqslant K_{V_i}(1+|\boldsymbol{x}|^{\beta_i}+|\boldsymbol{y}|^{\beta_i}).\quad(13)$$

令 $\beta=\max\{\beta_i\},i=1,2,3$.

(A4) 存在常数 $K_3>0,q>2$,使得对任意的 $\boldsymbol{x},\boldsymbol{y},\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}\in\mathbf{R}^n$,有:

$$\begin{aligned}&(\boldsymbol{x}-D(\boldsymbol{y})-\bar{\boldsymbol{x}}+D(\bar{\boldsymbol{y}}))^{\mathrm{T}}(f(\boldsymbol{x},\boldsymbol{y})-f(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}))+\\&\frac{q-1}{2}|g(\boldsymbol{x},\boldsymbol{y})-g(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})|^2\leqslant\\&K_3|\boldsymbol{x}-\bar{\boldsymbol{x}}|^2+|V_1(\boldsymbol{y},\bar{\boldsymbol{y}})||\boldsymbol{y}-\bar{\boldsymbol{y}}|^2.\end{aligned}\quad(14)$$

(A5)存在常数 $K_4>0$ 和 $p>q$,使得对任意的 $\boldsymbol{x},\boldsymbol{y}\in\mathbf{R}^n$,有:

$$\begin{aligned}&(\boldsymbol{x}-D(\boldsymbol{y}))^{\mathrm{T}}f(\boldsymbol{x},\boldsymbol{y})+\frac{p-1}{2}|g(\boldsymbol{x},\boldsymbol{y})|^2\leqslant\\&K_4(1+|\boldsymbol{x}|^2)+|V_2(\boldsymbol{y},0)||\boldsymbol{y}|^2.\end{aligned}\quad(15)$$

下面的引理证明了(1)解析解的有界性.

引理 1 假设(A2)—(A5)成立.具有初值条件(2)的中立型随机延迟微分方程(1)在 $t\in[-\tau,T]$ 上存在唯一解 $\boldsymbol{x}(t)$,且:

$$\sup_{-\tau\leqslant t\leqslant T}\mathbb{E}|\boldsymbol{x}(t)|^p<\infty,\forall T>0.$$

(A6) 存在常数 $K_5>0$ 和 $p>q$,使得对任意的 $\boldsymbol{x},\boldsymbol{y}\in\mathbf{R}^n$,有:

$$\begin{aligned}&(\boldsymbol{x}-D(\boldsymbol{y}))^{\mathrm{T}}f_\Delta(\boldsymbol{x},\boldsymbol{y})+\frac{p-1}{2}|g_\Delta(\boldsymbol{x},\boldsymbol{y})|^2\leqslant\\&K_5(1+|\boldsymbol{x}|^2)+|V_3(\boldsymbol{y},0)||\boldsymbol{y}|^2.\end{aligned}\quad(16)$$

注 1 当 $D(\cdot)=0$,由 (A5) 可知对任意的 $\boldsymbol{x},\boldsymbol{y}\in\mathbf{R}^n$,有:

$$\begin{aligned}&\boldsymbol{x}^{\mathrm{T}}f_\Delta(\boldsymbol{x},\boldsymbol{y})+\frac{p-1}{2}|g_\Delta(\boldsymbol{x},\boldsymbol{y})|^2\leqslant\\&\bar{K}_5(1+|\boldsymbol{x}|^2)+|V_2(\boldsymbol{y},0)|^2|\boldsymbol{y}|^2,\end{aligned}$$

其中 $\bar{K}_5=3K_4([1/\varphi(1)]\vee 1)$. 即当不存在中立项时,可以去掉(A6).

采用文献[2]中引理 2.3 的方法,可以得到下面的引理.

引理 2 假设(A3)成立.对任意的 $\boldsymbol{x},\boldsymbol{y},\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}\in\mathbf{R}^n$, $\Delta\in(0,1]$ 且 $K_{\varphi(\Delta)}\geqslant 1$,有:

$$|f_\Delta(\boldsymbol{x},\boldsymbol{y})-f_\Delta(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})|\vee|g_\Delta(\boldsymbol{x},\boldsymbol{y})-g_\Delta(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})|\leqslant 2K_{\varphi(\Delta)}(|\boldsymbol{x}-\bar{\boldsymbol{x}}|+|\boldsymbol{y}-\bar{\boldsymbol{y}}|),$$

$$(\boldsymbol{x}-D(\boldsymbol{y})-\bar{\boldsymbol{x}}+D(\bar{\boldsymbol{y}}))^{\mathrm{T}}(f_\Delta(\boldsymbol{x},\boldsymbol{y})-f_\Delta(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}))\leqslant 9K_{\varphi(\Delta)}^2(|\boldsymbol{x}-\bar{\boldsymbol{x}}|^2+|\boldsymbol{y}-\bar{\boldsymbol{y}}|^2).$$

由引理 2 可得:

$$|f_\Delta(\boldsymbol{x},\boldsymbol{y})|\vee|g_\Delta(\boldsymbol{x},\boldsymbol{y})|\leqslant 2K_{\varphi(\Delta)}(|\boldsymbol{x}|+|\boldsymbol{y}|)+(|f(0,0)|+|g(0,0)|).\tag{17}$$

此外,由文献[6]中的单调算子理论可知:当 $9K_{\varphi(\Delta)}^2\theta\Delta<1$ 成立时,对于给定的 $\boldsymbol{x}_\Delta(t_k)$,存在唯一的 $\boldsymbol{x}_\Delta(t_{k+1})$.由(5)得到 $\theta^2\Delta<1/81$.另外,在下文的引理 4 证明中,为了得到强收敛率,需要 $\theta\Delta^{3/4}<1/8$ 成立.记 $\Delta^*=1\wedge[1/((81\theta^2)\vee(16\theta^{4/3}))]$.在本文剩下的部分中,假设 $\Delta\in(0,\Delta^*)$ 和 $\theta\in(0,1]$.为了方便,定义:

$$\mathcal{K}(t)=\lfloor t/\Delta\rfloor\Delta,\quad \forall -\tau\leqslant t\leqslant T.\tag{18}$$

引理 3 假设(A3)成立.对任意的 $\Delta\in(0,\Delta^*)$ 和 $t\in[0,T]$,有:

$$\mathbb{E}(|\boldsymbol{y}_\Delta(t)-\bar{\boldsymbol{y}}_\Delta(t)|^{\bar{p}}|\mathcal{F}_{\mathcal{K}(t)})\leqslant C\Delta^{\frac{\bar{p}}{2}}K_{\varphi(\Delta)}^{\bar{p}}(1+|\bar{\boldsymbol{x}}_\Delta(s)|^{\bar{p}}+|\bar{\boldsymbol{x}}_\Delta(s-\tau)|^{\bar{p}}),\ \forall\bar{p}>0.$$

证明 由(10)和(11)可得:

$$\boldsymbol{y}_\Delta(t)-D(\bar{\boldsymbol{x}}_\Delta(t-\tau))=\boldsymbol{y}_\Delta(0)-D(\xi(-\tau))+\int_0^t f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))\,\mathrm{d}s+\int_0^t g_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))\,\mathrm{d}\boldsymbol{B}(s),\tag{19}$$

其中 $\boldsymbol{y}_\Delta(0)=\xi(0)-\theta f_\Delta(\xi(0),\xi(-\tau))\Delta$.对任意固定的 $\bar{p}\geqslant 2$ 和 $t\in[0,T]$,一定存在整数 $k\geqslant 0$ 使得 $t\in[t_k,t_{k+1})$.由假设(A3),Hölder 不等式和 BDG 不等式可得:

$$\begin{aligned}&\mathbb{E}(|\boldsymbol{y}_\Delta(t)-\bar{\boldsymbol{y}}_\Delta(t)|^{\bar{p}}|\mathcal{F}_{\mathcal{K}(t)})\leqslant\\
&C\,\mathbb{E}\Big(\Big|\int_{\mathcal{K}(t)}^t f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))\,\mathrm{d}s\Big|^{\bar{p}}\Big|\mathcal{F}_{\mathcal{K}(t)}\Big)+\\
&C\,\mathbb{E}\Big(\Big|\int_{\mathcal{K}(t)}^t g_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))\,\mathrm{d}\boldsymbol{B}(s)\Big|^{\bar{p}}\Big|\mathcal{F}_{\mathcal{K}(t)}\Big)\leqslant\\
&C\Delta^{\bar{p}-1}\,\mathbb{E}\Big(\int_{\mathcal{K}(t)}^t|f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|^{\bar{p}}\mathrm{d}s\Big|\mathcal{F}_{\mathcal{K}(t)}\Big)+\\
&C\Delta^{\frac{\bar{p}}{2}-1}\,\mathbb{E}\Big(\int_{\mathcal{K}(t)}^t|g_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|^{\bar{p}}\mathrm{d}s\Big|\mathcal{F}_{\mathcal{K}(t)}\Big)\leqslant\\
&C\Delta^{\frac{\bar{p}}{2}}K_{\varphi(\Delta)}^{\bar{p}}(1+|\bar{\boldsymbol{x}}_\Delta(s)|^{\bar{p}}+|\bar{\boldsymbol{x}}_\Delta(s-\tau)|^{\bar{p}}).\end{aligned}$$

当 $\bar{p}\in(0,2)$,由 Jensen 不等式可得:

$$\begin{aligned}&\mathbb{E}(|\boldsymbol{y}_\Delta(t)-\bar{\boldsymbol{y}}_\Delta(t)|^{\bar{p}}|\mathcal{F}_{\mathcal{K}(t)})\leqslant\\
&(\mathbb{E}|y_\Delta(t)-\bar{\boldsymbol{y}}_\Delta(t)|^2|\mathcal{F}_{\mathcal{K}(t)})^{\frac{\bar{p}}{2}}\leqslant\\
&C\Delta^{\frac{\bar{p}}{2}}K_{\varphi(\Delta)}^{\bar{p}}(1+|\bar{\boldsymbol{x}}_\Delta(s)|^{\bar{p}}+|\bar{\boldsymbol{x}}_\Delta(s-\tau)|^{\bar{p}}).\end{aligned}$$

证毕.

引理 4 假设(A2),(A3)和(A6)成立.可得:

$$\sup_{\Delta\in(0,\Delta^*)}\ \sup_{t\in[0,T]}\mathbb{E}|\boldsymbol{x}_\Delta(t)|^p\leqslant C,\quad\forall T>0.$$

证明 由 Itô 公式和(19)可得:

$$\begin{aligned}&\mathbb{E}|\boldsymbol{y}_\Delta(t)-D(\bar{\boldsymbol{x}}_\Delta(t-\tau))|^p-|\boldsymbol{y}_\Delta(0)-D(\xi(-\tau))|^p\leqslant\\
&\mathbb{E}\int_0^t p|\boldsymbol{y}_\Delta(s)-D(\bar{\boldsymbol{x}}_\Delta(s-\tau))|^{p-2}\cdot\\
&\Big[(\boldsymbol{y}_\Delta(s)-D(\bar{\boldsymbol{x}}_\Delta(s-\tau)))^{\mathrm{T}}f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))+\\
&\frac{p-1}{2}|g_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|^2\Big]\mathrm{d}s\leqslant\\
&\mathbb{E}\int_0^t p|\boldsymbol{y}_\Delta(s)-D(\bar{\boldsymbol{x}}_\Delta(s-\tau))|^{p-2}\cdot\\
&\Big[(\bar{\boldsymbol{x}}_\Delta(s)-D(\bar{\boldsymbol{x}}_\Delta(s-\tau)))^{\mathrm{T}}f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))+\\
&\frac{p-1}{2}|g_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|^2\Big]\mathrm{d}s+\\
&\mathbb{E}\int_0^t\Big[p|\boldsymbol{y}_\Delta(s)-D(\bar{\boldsymbol{x}}_\Delta(s-\tau))|^{p-2}\cdot\\
&(\boldsymbol{y}_\Delta(s)-\bar{\boldsymbol{x}}_\Delta(s))^{T}f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))\Big]\mathrm{d}s=:H_1+H_2.\end{aligned}\tag{20}$$

由假设(A2),(A6)和 Hölder 不等式,可得:

$$\begin{aligned}H_1\leqslant&\ \mathbb{E}\int_0^t[p|\boldsymbol{y}_\Delta(s)-D(\bar{\boldsymbol{x}}_\Delta(s-\tau))|^{p-2}\cdot\\
&\Big(K_5(1+|\bar{\boldsymbol{x}}_\Delta(s)|^2)+\\
&|V_3(\bar{\boldsymbol{x}}_\Delta(s-\tau),0)||\bar{\boldsymbol{x}}_\Delta(s-\tau)|^2)]\mathrm{d}s\leqslant\\
&C\,\mathbb{E}\int_0^t|\boldsymbol{y}_\Delta(s)-D(\bar{\boldsymbol{x}}_\Delta(s-\tau))|^p\mathrm{d}s+\\
&C\,\mathbb{E}\int_0^t(1+|\bar{\boldsymbol{x}}_\Delta(s)|^p)\,\mathrm{d}s+\\
&C\,\mathbb{E}\int_0^t|V_3(\bar{\boldsymbol{x}}_\Delta(s-\tau),0)|^{\frac{p}{2}}|\bar{\boldsymbol{x}}_\Delta(s-\tau)|^p\mathrm{d}s\leqslant\\
&C\,\mathbb{E}\int_0^t\Big(1+|\boldsymbol{x}_\Delta(s)|^p+K_{\varphi(\Delta)}^p\Delta^p(|x_\Delta(s)|^p+\\
&|\boldsymbol{x}_\Delta(s-\tau)|^p)+|\bar{\boldsymbol{x}}_\Delta(s)|^p+|\bar{\boldsymbol{x}}_\Delta(s-\tau)|^p\Big)\,\mathrm{d}s+\\
&C\,\mathbb{E}\int_0^t\Big(|V_3(\bar{\boldsymbol{x}}_\Delta(s-\tau),0)|^p+|\bar{\boldsymbol{x}}_\Delta(s-\tau)|^{2p}\Big)\mathrm{d}s\leqslant\\
&C\Big(1+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|\boldsymbol{x}_\Delta(r)|^p\mathrm{d}s\Big)+\end{aligned}$$

$$C\int_0^t \mathbb{E}|\bar{\boldsymbol{x}}_\Delta(s-\tau)|^{p\bar{\beta}}\mathrm{d}s, \tag{21}$$

其中 $\bar{\beta}=\beta\vee 2$.另外,

$$H_2 \leqslant C\,\mathbb{E}\int_0^t |\bar{\boldsymbol{y}}_\Delta(s)-D(\bar{\boldsymbol{x}}_\Delta(s-\tau))|^{p-2}$$
$$|\boldsymbol{y}_\Delta(s)-\bar{\boldsymbol{x}}_\Delta(s)||f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|\mathrm{d}s+$$
$$C\,\mathbb{E}\int_0^t |\boldsymbol{y}_\Delta(s)-\bar{\boldsymbol{y}}_\Delta(s)|^{p-2}|y_\Delta(s)-\bar{\boldsymbol{x}}_\Delta(s)|\cdot$$
$$|f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|\mathrm{d}s=:H_{21}+H_{22}.$$

由引理 2,引理 3,Young 不等式和(5),可得:

$$H_{21}\leqslant C\int_0^t \mathbb{E}[|\bar{\boldsymbol{y}}_\Delta(s)-D(\bar{\boldsymbol{x}}_\Delta(s-\tau))|^{p-2}\cdot$$
$$|f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|\ \mathbb{E}(|\boldsymbol{y}_\Delta(s)-\bar{\boldsymbol{y}}_\Delta(s)-\theta f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),$$
$$\bar{\boldsymbol{x}}_\Delta(s-\tau))\Delta\,|\,\mathcal{F}_{\mathcal{K}(s)})]\mathrm{d}s\leqslant$$
$$C\int_0^t \mathbb{E}[|\bar{\boldsymbol{y}}_\Delta(s)-D(\bar{\boldsymbol{x}}_\Delta(s-\tau))|^{p-2}\cdot$$
$$|f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|\mathbb{E}(|\boldsymbol{y}_\Delta(s)-\bar{\boldsymbol{y}}_\Delta(s)|\mathcal{F}_{\mathcal{K}(s)})]\mathrm{d}s+$$
$$C\int_0^t \mathbb{E}[|\bar{\boldsymbol{y}}_\Delta(s)-D(\bar{\boldsymbol{x}}_\Delta(s-\tau))|^{p-2}\cdot$$
$$|f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|^2\Delta]\mathrm{d}s\leqslant$$
$$C\int_0^t(\Delta^{\frac{1}{2}}K_{\varphi(\Delta)}^2+1)\,\mathbb{E}|\bar{\boldsymbol{y}}_\Delta(s)-D(\bar{\boldsymbol{x}}_\Delta(s-\tau))|^p\mathrm{d}s+$$
$$C\int_0^t\Delta^{\frac{1}{2}}K_{\varphi(\Delta)}^2\,\mathbb{E}(1+|\bar{\boldsymbol{x}}_\Delta(s)|^p+|\bar{\boldsymbol{x}}_\Delta(s-\tau)|^p)\mathrm{d}s+$$
$$C\int_0^t\Delta^{\frac{p}{2}}\,\mathbb{E}|f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|^p\mathrm{d}s\leqslant$$
$$C\left(1+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|x_\Delta(r)|^p\mathrm{d}s\right).$$

同理可得:

$$H_{22}\leqslant C\left(1+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|\boldsymbol{x}_\Delta(r)|^p\mathrm{d}s\right).$$

因此,

$$H_2\leqslant H_{21}+H_{22}\leqslant C\left(1+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|\boldsymbol{x}_\Delta(r)|^p\mathrm{d}s\right). \tag{22}$$

把(21)和(22)插入(20)可以得到:

$$\mathbb{E}|\boldsymbol{y}_\Delta(t)-D(\bar{\boldsymbol{x}}_\Delta(t-\tau))|^p\leqslant$$
$$C\left(1+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|\boldsymbol{x}_\Delta(r)|^p\mathrm{d}s+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|\bar{\boldsymbol{x}}_\Delta(r-\tau)|^{p\bar{\beta}}\mathrm{d}s\right).$$

所以,

$$\sup_{0\leqslant r\leqslant t}\mathbb{E}|\boldsymbol{y}_\Delta(r)-D(\bar{\boldsymbol{x}}_\Delta(r-\tau))|^p\leqslant$$
$$C\left(1+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|\boldsymbol{x}_\Delta(r)|^p\mathrm{d}s+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|\bar{\boldsymbol{x}}_\Delta(r-\tau)|^{p\bar{\beta}}\mathrm{d}s\right).$$

注意到对任意的 $\hat{c}>0$,有:

$$\sup_{0\leqslant r\leqslant t}\mathbb{E}|y_\Delta(r)|^p\leqslant$$
$$(1+\hat{c})^{p-1}\sup_{0\leqslant r\leqslant t}\mathbb{E}|y_\Delta(r)-D(\bar{\boldsymbol{x}}_\Delta(r-\tau))|^p+$$
$$\left(\frac{1+\hat{c}}{\hat{c}}\right)^{p-1}K_2^p(\|\xi\|^p+\sup_{0\leqslant r\leqslant t}\mathbb{E}|y_\Delta(r)|^p).$$

当 $\hat{c}$ 足够大时, 对任意的 $K_2\in(0,1)$, 有 $((1+\hat{c})/\hat{c})^{p-1}K_2^p<1$ 成立.因此,

$$\sup_{0\leqslant r\leqslant t}\mathbb{E}|\boldsymbol{y}_\Delta(r)|^p\leqslant$$
$$c_1\sup_{0\leqslant r\leqslant t}\mathbb{E}|\boldsymbol{y}_\Delta(r)-D(\bar{\boldsymbol{x}}_\Delta(r-\tau))|^p+c_2\|\xi\|^p,$$

其中 $c_1=\dfrac{\hat{c}^{p-1}(1+\hat{c})^{p-1}}{\hat{c}^{p-1}-(1+\hat{c})^{p-1}K_2^p}$ 和 $c_2=\dfrac{K_2^p(1+\hat{c})^{p-1}}{\hat{c}^{p-1}-(1+\hat{c})^{p-1}K_2^p}$.

另外,由引理 2 和不等式 $|a-b|^p\geqslant 2^{1-p}|a|^p-|b|^p$ 可得:

$$|\boldsymbol{y}_\Delta(t)|^p\geqslant 2^{1-p}|\boldsymbol{x}_\Delta(t)|^p-$$
$$\theta^p\Delta^p|f_\Delta(\boldsymbol{x}_\Delta(t),\boldsymbol{x}_\Delta(t-\tau))|^p\geqslant 2^{1-p}|\boldsymbol{x}_\Delta(t)|^p-$$
$$2^{2p-2}\theta^p\Delta^pK_{\varphi(\Delta)}^p(|\boldsymbol{x}_\Delta(t)|^p+|\boldsymbol{x}_\Delta(t-\tau)|^p)-$$
$$2^{p-1}\theta^p\Delta^p(|f(0,0)|+|g(0,0)|)=$$
$$(2^{1-p}-2^{2p-2}\theta^p\Delta^pK_{\varphi(\Delta)}^p)|x_\Delta(t)|^p-$$
$$2^{2p-2}\theta^p\Delta^pK_{\varphi(\Delta)}^p|\boldsymbol{x}_\Delta(t-\tau)|^p-$$
$$2^{p-1}\theta^p\Delta^p(|f(0,0)|+|g(0,0)|).$$

已知 $\Delta<1/(16\theta^{4/3})$,所以有:

$$\sup_{0\leqslant r\leqslant t}\mathbb{E}|\boldsymbol{x}_\Delta(r)|^p\leqslant$$
$$C\left(1+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|\boldsymbol{y}_\Delta(r)|^p\mathrm{d}s\right)\leqslant$$
$$C\left(1+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|y_\Delta(r)-D(\bar{\boldsymbol{x}}_\Delta(r-\tau))|^p\mathrm{d}s\right)\leqslant$$
$$C\left(1+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|\boldsymbol{x}_\Delta(r)|^p\mathrm{d}s+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|\bar{\boldsymbol{x}}_\Delta(r-\tau)|^{p\bar{\beta}}\mathrm{d}s\right).$$

由 Gronwall 不等式可得:

$$\sup_{0\leqslant r\leqslant t}\mathbb{E}|x_\Delta(r)|^p\leqslant$$
$$C\left(1+\int_0^t\sup_{0\leqslant r\leqslant s}\mathbb{E}|\bar{\boldsymbol{x}}_\Delta(r-\tau)|^{p\bar{\beta}}\mathrm{d}s\right). \tag{23}$$

定义 $p_i=(\lfloor T/\tau\rfloor+2-i)p\bar{\beta}^{\lfloor T/\tau\rfloor+1-i}, i=1,2,\cdots,\dfrac{M'}{M}+1$.注意到 $p_{i+1}\bar{\beta}<p_i$ 和 $p_{\lfloor T/\tau\rfloor+1}=p, i=1,2,\cdots,\dfrac{M'}{M}$.

由 ξ 的定义和(23) 可得:

$$\sup_{0\leqslant r\leqslant\tau}\mathbb{E}|\boldsymbol{x}_\Delta(r)|p_1\leqslant C.$$

再由 Hölder 不等式可得:

$$\sup_{0\leqslant r\leqslant 2\tau}\mathbb{E}|\boldsymbol{x}_\Delta(r)|p_2\leqslant$$
$$C\left(1+\int_0^{2\tau}\sup_{0\leqslant r\leqslant s}\mathbb{E}(|\boldsymbol{x}_\Delta(r-\tau)|^{p_1})^{\frac{p_2\bar{\beta}}{p_1}}\mathrm{d}s\right)\leqslant C.$$

重复此过程可以得到所需结论,证毕.

利用条件期望的性质和类似引理 4 的方法可以得到下面的引理.

引理 5 假设(A2),(A3)和(A6)成立.对任意

的 $\Delta \in (0,\Delta^*)$ 和 $t \in [0,T]$,有:

$$\mathbb{E}|\boldsymbol{y}_\Delta(t) - \bar{\boldsymbol{y}}_\Delta(t)|^p \leqslant C\Delta^{\frac{p}{2}}K^p_{\varphi(\Delta)},$$

$$\mathbb{E}|\boldsymbol{x}_\Delta(t) - \bar{\boldsymbol{x}}_\Delta(t)|^p \leqslant C\Delta^{\frac{p}{2}}K^p_{\varphi(\Delta)}.$$

因此,

$$\lim_{\Delta\to 0}\mathbb{E}|\boldsymbol{y}_\Delta(t) - \bar{\boldsymbol{y}}_\Delta(t)|^p = \lim_{\Delta\to 0}\mathbb{E}|\boldsymbol{x}_\Delta(t) - \bar{\boldsymbol{x}}_\Delta(t)|^p = 0.$$

此外,由引理1,引理4和Chebyshev不等式可以得到下面的引理.

引理6 假设(A2)—(A6)成立.对任意的 $R \geqslant \|\xi\|$,定义停时:

$$\tau_R = \inf\{t \geqslant 0: |\boldsymbol{x}(t)| \geqslant R\},$$

$$\tau_{\Delta,R} = \inf\{t \geqslant 0: |\boldsymbol{x}_\Delta(t)| \geqslant R\}.$$

则有: $\mathbb{P}(\tau_R \leqslant T) \leqslant \dfrac{C}{R^p}$ 和 $\mathbb{P}(\tau_{\Delta,R} \leqslant T) \leqslant \dfrac{C}{R^p}$.

引理7 假设(A1)—(A6)成立.令 $\Delta \in (0,\Delta^*)$ 充分小使得 $\varphi(\Delta) \geqslant R \vee \|\xi\|$ 成立.可得:

$$\mathbb{E}|\boldsymbol{x}(T\wedge\rho_{\Delta,R}) - \boldsymbol{x}_\Delta(T\wedge\rho_{\Delta,R})|^q \leqslant C\Delta^{\frac{q}{2}}K^{2q}_{\varphi(\Delta)} \vee \Delta^{\gamma q},$$

其中 $\rho_{\Delta,R} := \tau_R \wedge \tau_{\Delta,R}$.

证明 令 $\boldsymbol{e}_\Delta(t) = \boldsymbol{x}(t) - D(\boldsymbol{x}(t-\tau)) - y_\Delta(t) + D(\bar{\boldsymbol{x}}_\Delta(t-\tau))$.为了方便,记 $\rho_{\Delta,L} = \rho$.显然,对于 $0 \leqslant s \leqslant t \wedge \rho$,有 $|\boldsymbol{x}(s)| \vee |\boldsymbol{x}(s-\tau)| \vee |\bar{\boldsymbol{x}}_\Delta(s)| \vee |\bar{\boldsymbol{x}}_\Delta(s-\tau)| \leqslant R \leqslant \varphi(\Delta)$.

因此,对于 $0 \leqslant s \leqslant t\wedge\rho$,有:

$$f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau)) = f(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau)),$$

$$g_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau)) = g(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau)).$$

由Itô公式得到:

$$\begin{aligned}
&\mathbb{E}|\boldsymbol{e}_\Delta(t\wedge\rho)|^q - \theta^q|f_\Delta(\xi(0),\xi(-\tau))|^q\Delta^q \leqslant \\
&\mathbb{E}\int_0^{t\wedge\rho} q\,|\boldsymbol{e}_\Delta(s)|q-2\Big[\boldsymbol{e}^{\mathrm{T}}_\Delta(s)\,(f(\boldsymbol{x}(s),\boldsymbol{x}(s-\tau)) - \\
&f_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau)))+\frac{q-1}{2}|g(\boldsymbol{x}(s),\boldsymbol{x}(s-\tau)) - \\
&g_\Delta(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|^2\Big]\mathrm{d}s \leqslant \mathbb{E}\int_0^{t\wedge\rho} q|\boldsymbol{e}_\Delta(s)|^{q-2}\cdot \\
&\Big[(\boldsymbol{x}(s) - D(\boldsymbol{x}(s-\tau)) - \bar{\boldsymbol{x}}_\Delta(s) + \\
&D(\bar{\boldsymbol{x}}_\Delta(s-\tau)))^{\mathrm{T}}\cdot(f(\boldsymbol{x}(s),\boldsymbol{x}(s-\tau)) - \\
&f(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau)))+\frac{q-1}{2}|g(\boldsymbol{x}(s),\boldsymbol{x}(s-\tau)) - \\
&g(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|^2\Big]\mathrm{d}s + \\
&\mathbb{E}\int_0^{t\wedge\rho} q|\boldsymbol{e}_\Delta(s)|^{q-2}(\bar{\boldsymbol{x}}_\Delta(s) - \boldsymbol{x}_\Delta(s))^{\mathrm{T}} \\
&(f(\boldsymbol{x}(s),\boldsymbol{x}(s-\tau)) - f(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau)))\mathrm{d}s + \\
&\mathbb{E}\int_0^{t\wedge\rho} q|\boldsymbol{e}_\Delta(s)|^{q-2}(\boldsymbol{x}_\Delta(s) - \boldsymbol{y}_\Delta(s))^{\mathrm{T}}\cdot \\
&(f(\boldsymbol{x}(s),\boldsymbol{x}(s-\tau)) - f(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau)))\mathrm{d}s =: \\
&J_1 + J_2 + J_3.
\end{aligned}$$

由(A4), Young不等式,Hölder不等式,引理4和引理5可得:

$$\begin{aligned}
J_1 \leqslant\ & C\Big(\mathbb{E}\int_0^{t\wedge\rho}|\boldsymbol{e}_\Delta(s)|^q\mathrm{d}s + \mathbb{E}\int_0^{t\wedge\rho}|\boldsymbol{x}(s) - \bar{\boldsymbol{x}}_\Delta(s)|^q\mathrm{d}s + \\
&\mathbb{E}\int_0^{t\wedge\rho}|V_1(\boldsymbol{x}(s-\tau),\bar{\boldsymbol{x}}_\Delta(s-\tau))|^{\frac{q}{2}}\cdot \\
&|\boldsymbol{x}(s-\tau) - \bar{\boldsymbol{x}}_\Delta(s-\tau)|^q\mathrm{d}s) \leqslant \\
&C\Big(\int_0^t \mathbb{E}|\boldsymbol{e}_\Delta(s\wedge\rho)|^q\mathrm{d}s + \\
&\int_0^t \mathbb{E}|\boldsymbol{x}(s\wedge\rho) - \boldsymbol{x}_\Delta(s\wedge\rho)|^q\mathrm{d}s + \\
&\int_0^T \mathbb{E}|\boldsymbol{x}_\Delta(s) - \bar{\boldsymbol{x}}_\Delta(s)|^q\mathrm{d}s + \\
&\int_0^T(\mathbb{E}|V_1(\boldsymbol{x}(s-\tau),\bar{\boldsymbol{x}}_\Delta(s-\tau))|^q)^{\frac{1}{2}}\cdot \\
&(\mathbb{E}|x_\Delta(s-\tau) - \bar{\boldsymbol{x}}_\Delta(s-\tau)|^{2q})^{\frac{1}{2}}\mathrm{d}s + \\
&\mathbb{E}\int_0^{t\wedge\rho}|V_1(\boldsymbol{x}(s-\tau),\bar{\boldsymbol{x}}_\Delta(s-\tau))|^{\frac{q}{2}}\cdot \\
&|\boldsymbol{x}(s-\tau) - \boldsymbol{x}_\Delta(s-\tau)|^q\mathrm{d}s) \leqslant \\
&C\Big(\int_0^t \mathbb{E}|\boldsymbol{e}_\Delta(s\wedge\rho)|^q\mathrm{d}s + \\
&\int_0^t \mathbb{E}|\boldsymbol{x}(s\wedge\rho) - \boldsymbol{x}_\Delta(s\wedge\rho)|^q\mathrm{d}s + \Delta^{\frac{q}{2}}K^q_{\varphi(\Delta)} + \\
&\int_0^t(\mathbb{E}|V_1(\boldsymbol{x}(s\wedge\rho-\tau),\bar{\boldsymbol{x}}_\Delta(s\wedge\rho-\tau))|^q)^{\frac{1}{2}}\cdot \\
&(\mathbb{E}|\boldsymbol{x}(s\wedge\rho-\tau) - \boldsymbol{x}_\Delta(s\wedge\rho-\tau)|^{2q})^{\frac{1}{2}}\mathrm{d}s) \leqslant \\
&C(\int_0^t \mathbb{E}|\boldsymbol{e}_\Delta(s\wedge\rho)|^q\mathrm{d}s + \\
&\int_0^t \mathbb{E}|\boldsymbol{x}(s\wedge\rho) - \boldsymbol{x}_\Delta(s\wedge\rho)|^q\mathrm{d}s + \Delta^{\frac{q}{2}}K^q_{\varphi(\Delta)} + \\
&\int_0^t(\mathbb{E}|\boldsymbol{x}(s\wedge\rho-\tau) - \boldsymbol{x}_\Delta(s\wedge\rho-\tau)|^{2q})^{\frac{1}{2}}\mathrm{d}s).
\end{aligned}$$

由引理2,引理5和Young不等式可得:

$$\begin{aligned}
J_2 \leqslant\ & C\Big(\mathbb{E}\int_0^{t\wedge\rho}|\boldsymbol{e}_\Delta(s)|^q\mathrm{d}s + \\
&\mathbb{E}\int_0^{t\wedge\rho}K^q_{\varphi(\Delta)}|\boldsymbol{x}_\Delta(s) - \bar{\boldsymbol{x}}_\Delta(s)|^q\mathrm{d}s + \\
&\mathbb{E}\int_0^{t\wedge\rho}\Big(|\boldsymbol{x}(s) - \bar{\boldsymbol{x}}_\Delta(s)|^q + \\
&|x(s-\tau) - \bar{\boldsymbol{x}}_\Delta(s-\tau)|^q\Big)\mathrm{d}s\Big) \leqslant \\
&C\Big(\mathbb{E}\int_0^{t\wedge\rho}|\boldsymbol{e}_\Delta(s)|^q\mathrm{d}s + \\
&\mathbb{E}\int_0^{t\wedge\rho}K^q_{\varphi(\Delta)}|\boldsymbol{x}_\Delta(s) - \bar{\boldsymbol{x}}_\Delta(s)|^q\mathrm{d}s + \\
&\mathbb{E}\int_0^{t\wedge\rho}|\boldsymbol{x}(s) - \boldsymbol{x}_\Delta(s)|^q\mathrm{d}s +
\end{aligned}$$

$$\mathbb{E}\int_0^{t\wedge\rho}|\boldsymbol{x}_\Delta(s)-\bar{\boldsymbol{x}}_\Delta(s)|^q\mathrm{d}s+\mathbb{E}\int_{-\tau}^0|\xi(s)-\xi(K(\mathrm{s}))|^q\mathrm{d}s)\leqslant$$
$$C\Big(\int_0^t\mathbb{E}|\boldsymbol{e}_\Delta(s\wedge\rho)|^q\mathrm{d}s+\int_0^t\mathbb{E}|\boldsymbol{x}(s\wedge\rho)-\boldsymbol{x}_\Delta(s\wedge\rho)|^q\mathrm{d}s+\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q}+\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{q}+\Delta^{\gamma q}\Big).$$

同理可得:

$$J_3\leqslant C(\mathbb{E}\int_0^{t\wedge\rho}|\boldsymbol{e}_\Delta(s)|^q\mathrm{d}s+\mathbb{E}\int_0^{t\wedge\rho}(\theta\Delta|f_\Delta(\boldsymbol{x}_\Delta(s),\boldsymbol{x}_\Delta(s-\tau))|\cdot|f(\boldsymbol{x}(s),\boldsymbol{x}(s-\tau))-f(\bar{\boldsymbol{x}}_\Delta(s),\bar{\boldsymbol{x}}_\Delta(s-\tau))|)^{\frac{q}{2}}\mathrm{d}s)\leqslant$$
$$C\Big(\int_0^t\mathbb{E}|\boldsymbol{e}_\Delta(s\wedge\rho)|^q\mathrm{d}s+\int_0^t\mathbb{E}|\boldsymbol{x}(s\wedge\rho)-\boldsymbol{x}_\Delta(s\wedge\rho)|^q\mathrm{d}s+\Delta^{q}K_{\varphi(\Delta)}^{2q}+\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{q}+\Delta^{\gamma q}\Big).$$

将这些结果结合到一起可得:

$$\mathbb{E}|\boldsymbol{e}_\Delta(t\wedge\rho)|^q\leqslant C\Big(\int_0^t\mathbb{E}|\boldsymbol{e}_\Delta(s\wedge\rho)|^q\mathrm{d}s+\int_0^t\mathbb{E}|\boldsymbol{x}(s\wedge\rho)-\boldsymbol{x}_\Delta(s\wedge\rho)|^q\mathrm{d}s+\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q}+\Delta^{\gamma q}+\int_0^t(\mathbb{E}|\boldsymbol{x}(s\wedge\rho-\tau)-\boldsymbol{x}_\Delta(s\wedge\rho-\tau)|^{2q})^{\frac{1}{2}}\mathrm{d}s\Big).$$

由 Gronwall 不等式可得:

$$\mathbb{E}|\boldsymbol{e}_\Delta(t\wedge\rho)|^q\leqslant C\Big(\int_0^t\mathbb{E}|\boldsymbol{x}(s\wedge\rho)-\boldsymbol{x}_\Delta(s\wedge\rho)|^q\mathrm{d}s+\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q}+\Delta^{\gamma q}+\int_0^t(\mathbb{E}|\boldsymbol{x}(s\wedge\rho-\tau)-\boldsymbol{x}_\Delta(s\wedge\rho-\tau)|^{2q})^{\frac{1}{2}}\mathrm{d}s\Big).$$

注意到对任意的 $t\in[0,T]$,有:

$$\mathbb{E}|\boldsymbol{x}(t\wedge\rho)-\boldsymbol{x}_\Delta(t\wedge\rho)|^q\leqslant 3^{q-1}\mathbb{E}|\boldsymbol{e}_\Delta(t\wedge\rho)|^q+3^{q-1}\mathbb{E}|\theta f_\Delta(\boldsymbol{x}_\Delta(t\wedge\rho),\boldsymbol{x}_\Delta(t\wedge\rho-\tau))\Delta|^q+3^{q-1}\mathbb{E}|D(\boldsymbol{x}(t\wedge\rho-\tau))-D(\bar{\boldsymbol{x}}_\Delta(t\wedge\rho-\tau))|^q\leqslant$$
$$C\Big(\int_0^t\mathbb{E}|\boldsymbol{x}(s\wedge\rho)-\boldsymbol{x}_\Delta(s\wedge\rho)|^q\mathrm{d}s+\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q}+\Delta^{\gamma q}+\int_0^t(\mathbb{E}|\boldsymbol{x}(s\wedge\rho-\tau)-\boldsymbol{x}_\Delta(s\wedge\rho-\tau)|^{2q})^{\frac{1}{2}}\mathrm{d}s\Big).$$

由 Gronwall 不等式可得:

$$\mathbb{E}|\boldsymbol{x}(t\wedge\rho)-\boldsymbol{x}_\Delta(t\wedge\rho)|^q\leqslant C\Big(\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q}+\Delta^{\gamma q}+\int_0^t(\mathbb{E}|\boldsymbol{x}(s\wedge\rho-\tau)-\boldsymbol{x}_\Delta(s\wedge\rho-\tau)|^{2q})^{\frac{1}{2}}\mathrm{d}s\Big).$$

定义 $q_i=(\lfloor T/\tau\rfloor+2-i)q2^{\lfloor T/\tau\rfloor+1-i}$, $i=1,2,\cdots,\frac{M'}{M}+1$.注意到 $2q_{i+1}<q_i$ 和 $q_{\lfloor T/\tau\rfloor+1}=q$, $i=1,2,\cdots,\frac{M'}{M}$.

由 $|\boldsymbol{x}(s-\tau)-\boldsymbol{x}_\Delta(s-\tau)|=0$, $0\leqslant s\leqslant\tau$ 可得:

$$\sup_{0\leqslant s\leqslant\tau}\mathbb{E}|\boldsymbol{x}(s\wedge\rho)-\boldsymbol{x}_\Delta(s\wedge\rho)|^{q_1}\leqslant C(\Delta^{\frac{q_1}{2}}K_{\varphi(\Delta)}^{2q_1}\vee\Delta^{\gamma q_1}).$$

由 Hölder 不等式可得:

$$\sup_{0\leqslant s\leqslant 2\tau}\mathbb{E}|\boldsymbol{x}(s\wedge\rho)-\boldsymbol{x}_\Delta(s\wedge\rho)|^{q_2}\leqslant C\Big(\Delta^{\frac{q_2}{2}}K_{\varphi(\Delta)}^{2q_2}\vee\Delta^{\gamma q_2}+\int_0^{2\tau}(\mathbb{E}|\boldsymbol{x}(s\wedge\rho-\tau)-\boldsymbol{x}_\Delta(s\wedge\rho-\tau)|^{2q_2\frac{q_1}{2q_2}})^{\frac{q_2}{q_1}}\mathrm{d}s\Big)\leqslant C(\Delta^{\frac{q_2}{2}}K_{\varphi(\Delta)}^{2q_2}\vee\Delta^{\gamma q_2}).$$

重复此过程可以得到所需结论.证毕.

定理 1 假设(A1)—(A6)成立.对任意充分小的 $\Delta\in(0,\Delta^*)$,设:

$$\varphi(\Delta)\geqslant\tilde{c}\Big(\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q}\vee\Delta^{\gamma q}\Big)^{\frac{-1}{p-q}},\tag{24}$$

这里的 $\tilde{c}$ 是正常数,则有:

$$\mathbb{E}|\boldsymbol{x}(T)-\boldsymbol{x}_\Delta(T)|^q\leqslant C(\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q}\vee\Delta^{\gamma q}),\tag{25}$$
$$\mathbb{E}|\boldsymbol{x}(T)-\bar{\boldsymbol{x}}_\Delta(T)|^q\leqslant C(\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q}\vee\Delta^{\gamma q}).\tag{26}$$

证明 设 $\delta>0$ 是任意的实数.对任意的 $a,b>0$,由 Young 不等式可得:

$$a^qb=(\delta a^p)\frac{q}{p}\left(\frac{b^{p/(p-q)}}{\delta^{q/(p-q)}}\right)^{\frac{p-q}{p}}\leqslant\frac{q\delta}{p}a^p+\frac{p-q}{p\delta^{q/(p-q)}}b^{p/(p-q)}.$$

因此,由引理 1,引理 4 和引理 6 可得:

$$\mathbb{E}|\boldsymbol{x}(T)-\boldsymbol{x}_\Delta(T)|^q=\mathbb{E}(|\boldsymbol{x}(T)-\boldsymbol{x}_\Delta(T)|^q\mathbb{I}_{\{\rho>T\}})+\mathbb{E}(|\boldsymbol{x}(T)-\boldsymbol{x}_\Delta(T)|^q\mathbb{I}_{\{\rho\leqslant T\}})\leqslant$$
$$\mathbb{E}(|\boldsymbol{x}(T)-\boldsymbol{x}_\Delta(T)|^q\mathbb{I}_{\{\rho>T\}})+\frac{q\delta}{p}\mathbb{E}|\boldsymbol{x}(T)-\boldsymbol{x}_\Delta(T)|^p+\frac{p-q}{p\delta^{q/(p-q)}}\mathbb{P}(\rho\leqslant T)\leqslant$$
$$\mathbb{E}|\boldsymbol{x}(T\wedge\rho)-\boldsymbol{x}_\Delta(T\wedge\rho)|^q+\frac{Cq\delta}{p}+\frac{C(p-q)}{p\delta^{q/(p-q)}R^p}.$$

取 $\delta=\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q}\vee\Delta^{\gamma q}$ 和 $R=\tilde{c}\Big(\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q}\vee\Delta^{\gamma q}\Big)^{\frac{-1}{p-q}}$,可得:

$$\mathbb{E}|\boldsymbol{x}(T)-\boldsymbol{x}_\Delta(T)|^q\leqslant\mathbb{E}|\boldsymbol{x}(T\wedge\rho)-\boldsymbol{x}_\Delta(T\wedge\rho)|^q+C\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q}\vee\Delta^{\gamma q}.$$

由条件(24)得到:

$$\varphi(\Delta) \geqslant \tilde{c}(\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q} \vee \Delta^{\gamma q})^{\frac{-1}{p-q}} = R.$$

由引理7可得:

$$\mathbb{E}|\boldsymbol{x}(T) - \boldsymbol{x}_\Delta(T)|^q \leqslant C(\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q} \vee \Delta^{\gamma q}).$$

此外,由引理5和式(25)可以得到式(26).证毕.

3 实例

考虑一维的非线性中立型随机延迟微分方程:

$$\mathrm{d}\left[x(t) + \frac{1}{12}x(t-\tau)\right] = [-4x^5(t-\tau) - 20x(t) + 2x(t-\tau)]\mathrm{d}t + |x(t-\tau)|^{\frac{5}{2}}\mathrm{d}B(t),$$

且其初值 x_0 满足假设(A1).其中,$B(t)$ 是一维 Brownian 运动.显然,假设(A2)—(A6)成立.其中,有:$K_2 = 1/12, L_R = 4R^4, K_3 = 60, K_4 = K_5 = 260$, $V_1(y,\bar{y}) = 40 + ((q-1) \vee 40)(1 + |y|^8 + |\bar{y}|^8)$ 和 $V_2(y,0) = V_3(y,0) = 240 + ((p-1) \vee 40) \cdot (1 + |y|^8)$.

对于 $\varepsilon \in (0,1/4]$,定义 $\varphi(\Delta) = \sqrt[4]{1/4}\Delta^{-\frac{\varepsilon}{4}}$.因此,当 $K_{\varphi(\Delta)} = 4(\varphi(\Delta))^4$ 时,可以得到 $K_{\varphi(\Delta)}\Delta^{\frac{1}{4}} = \Delta^{-\varepsilon+\frac{1}{4}} \leqslant 1$ 和 $\lim\limits_{\Delta\to 0}\varphi(\Delta) = \infty$.选取 $q = 2, p = 3, \tilde{c} = \sqrt[4]{1/4}, \varepsilon = 33/136$.可得:

$$\varphi(\Delta) = \sqrt[4]{\frac{1}{4}}\Delta^{-\frac{\varepsilon}{4}} \geqslant \sqrt[4]{\frac{1}{4}}\Delta^{4\varepsilon-1} = \sqrt[4]{\frac{1}{4}}(\Delta^{\frac{q}{2}}K_{\varphi(\Delta)}^{2q} \vee \Delta^{\gamma q})^{\frac{-1}{p-q}}.$$

当 $q = 2$ 时,由定理1可得:

$$\mathbb{E}|x(T) - x_\Delta(T)|^2 \leqslant C\Delta^{(1-4\varepsilon)\wedge 2\gamma}.$$

这个例子说明了此算法的有效性,即本文的结论覆盖了一类非线性中立型随机延迟微分方程.

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The truncated θ-EM method for a class of nonlinear neutral stochastic delay differential equations

LI Yan[1] WANG Zhaohang[2] GAO Shuaibin[3]
1 College of Science, Huazhong Agricultural University, Wuhan 430070
2 School of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074
3 Mathematics & Science College of Shanghai Normal University, Shanghai 200233

Abstract Here we consider a class of nonlinear neutral stochastic delay differential equations. The coefficients of the drift term and diffusion term could increase superlinearly, and the neutral term satisfies the contractive mapping condition. The truncated θ-EM method for this type of equations is established and the convergence rate is obtained. Finally, an example is given to verify the theoretical result.

Key words stochastic delay differential equations; neutral term; truncated θ-EM algorithm; strong convergence rate