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王廷秀[1]

一类非线性 Volterra 积分不等式

摘要

本文简要讨论 Gronwall 不等式的研究进展,并给出关于如下的一类非线性 Volterra 积分不等式的一个结果:

$$w(u(t)) \leqslant g(t) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(t,s) \prod_{j=1}^{m} H_{ij}(u(s)) G_{ij}(\max_{s-h \leqslant \xi \leqslant s} u(\xi)) \mathrm{d}s.$$

关键词

非线性积分不等式;Gronwall 不等式;Gronwall 类不等式;Volterra 积分微分不等式;Lyapunov 第二方法

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作者简介

王廷秀,男,博士,教授,主要研究方向为微分方程稳定性.tingxiu.wang@ tamuc.edu

1 美国德克萨斯农工大学康莫斯分校 数学系,康莫斯,德州,75428

0 关于 Gronwall 不等式

1919 年,在研究一个带参变量的微分方程系统时,Gronwall[1] 给出如下的著名引理:

定理 1(Gronwall 原始不等式) 对 $x_0 \leqslant x \leqslant x_0 + h$,连续函数 $z = z(x)$ 满足不等式:

$$0 \leqslant z \leqslant \int_{x_0}^{x} [Mz + A] \mathrm{d}x,$$

其中常数 M 和 A 非负,那么

$$0 \leqslant z \leqslant Ah\mathrm{e}^{Mh}, \quad x_0 \leqslant x \leqslant x_0 + h.$$

在以后的 24 年里,Gronwall 原始不等式都没引起关注.1943 年,Bellman[2] 推广了 Gronwall 原始不等式使得 M 可以是函数,并且不等式也被陈述得更为简单、明了.这个结果被称之为 Gronwall-Bellman 不等式,在许多文献中均可查到,如文献[2-6].

定理 2(Gronwall-Bellman 不等式) 已知 $u(t)$ 和 $f(t)$ 是定义在 $[a,b]$ 区间上的非负连续函数,c 是非负常数,如果

$$u(t) \leqslant c + \int_{a}^{t} f(s) u(s) \mathrm{d}s, \quad a \leqslant t \leqslant b,$$

那么

$$u(t) \leqslant c\mathrm{e}^{\int_a^t f(s)\mathrm{d}s}, \quad a \leqslant t \leqslant b.$$

1958 年,Bellman[7] 进一步改进了上述定理,使得 c 可以是一个非负非增连续函数.

定理 3(Bellman 不等式) 如果 $y(t)$ 是正的,且单调增,$x(t)$,$z(t) \geqslant 0$,那么

$$x(t) \leqslant y(t) + \int_{\alpha}^{t} x(s) z(s) \mathrm{d}s,$$

蕴含着

$$x(t) \leqslant y(t) \mathrm{e}^{\int_\alpha^t z(s)\mathrm{d}s}, \quad \alpha \leqslant t \leqslant \beta.$$

今天看,以上的 3 个定理都较为粗糙,因为定理的陈述不够完整、条件还可以改进.作为对前面 3 个定理的统一推广,1966 年,Halanay 在专著[8]中,给出了下面的定理 4.这个定理被广泛引用,如文献[9].其条件比以上 3 个定理都要弱一些.这里,我们使用 Hale-Lunel [1993,p15]的陈述.

定理 4 如果 u 和 α 是定义在$[a,b]$区间上的实值连续函数,

$\beta \geqslant 0$ 在$[a,b]$区间上可积,且满足

$$u(t) \leqslant \alpha(t) + \int_a^t \beta(s)u(s)\mathrm{d}s, \quad a \leqslant t \leqslant b,$$

那么

$$u(t) \leqslant \alpha(t) + \int_a^t \beta(s)\alpha(s)\mathrm{e}^{\int_s^t \beta(u)\mathrm{d}u}\mathrm{d}s, \quad a \leqslant t \leqslant b.$$

此外,如果 α 非减,那么

$$u(t) \leqslant \alpha(t)\mathrm{e}^{\int_a^t \beta(s)\mathrm{d}s}, \quad a \leqslant t \leqslant b.$$

不像前面 3 个定理,定理 4 只要求 β 非负.当然,如果 u 是非负的,α 也相应必须非负.Gronwall 不等式在微分方程有界性、稳定性、存在性及其他定性性质的研究中有了大量、广泛的应用,对 Gronwall 不等式的应用、推广、研究爆发性增长,并产生了许多新的研究方向.1998 年出版的 Pachpatte 等的专著[6],收集、总结了在此之前对 Gronwall 不等式的研究、推广、应用.在众多推广中,本文讨论下面的推广.2000 年,Lipovan[10]研究了

$$u(t) \leqslant k + \int_{\alpha(t_0)}^{\alpha(t)} f(s)w(u(s))\mathrm{d}s.$$

2012 年,Bohner 等[11]研究了下面的不等式:

$$\psi(u(t)) \leqslant k + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)u^p(s)\omega_i(u(s))\mathrm{d}s + \sum_{j=1}^{m}\int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)u^p(s)\tilde{\omega}_j(\max_{\xi\in[s-h,s]} u(\xi))\mathrm{d}s. \quad (1)$$

2015 年,Wang[12]推广了不等式(1),用更一般的复合函数 $H_{ij}(u(s))$ 取代 $u^p(s)$,用

$$f_i(s)u^p(s)\omega_i(u(s))\tilde{\omega}_j(\max_{\xi\in[s-h,s]} u(\xi))$$

合并了两个级数,研究了下面的不等式:

$$w(u(t)) \leqslant K + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\prod_{j=1}^{m} H_{ij}(u(s))G_{ij}(\max_{s-h\leqslant\xi\leqslant s} u(\xi))\mathrm{d}s. \quad (2)$$

本文对不等式(2)进一步加以推广,使得 K 可以是函数,$f_i(s)$ 可以是 $f_i(t,s)$.从而,我们研究 Volterra 不等式:

$$w(u(t)) \leqslant g(t) + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(t,s)\prod_{j=1}^{m} H_{ij}(u(s))G_{ij}(\max_{s-h\leqslant\xi\leqslant s} u(\xi))\mathrm{d}s. \quad (3)$$

1 一类非线性 Volterra 积分不等式

我们研究不等式(3),并得到一个结果.由于不等式(3)涉及 7 类函数,为此,我们需要如下 6 个条件和记号:已知 $h > 0, t_0, T$ 为常数,$0 \leqslant t_0 < T \leqslant \infty$,$\mathbf{R}_+ = [0,\infty)$.

(A1) $g(t):[t_0,T) \to [0,\infty)$ 为连续非减函数;

(A2) $\alpha_i \in C^1([t_0,T),\mathbf{R}_+)$ 非减,并且 $\alpha_i(t) \leqslant t, t \in [t_0,T), i = 1,2,\cdots,n$;

(A3) $f_i(t,s) \in C([t_0,T)\times[t_0,T),\mathbf{R})$ 对 t 为连续非减函数,$i = 1,2,\cdots,n$;

(A4) $H_{ij}, G_{ij} \in C(\mathbf{R}_+,\mathbf{R}_+)$ 非减,且当 $x > 0$,$H_{ij}(x) > 0, G_{ij}(x) > 0$;

(A5) $w \in C(\mathbf{R}_+,\mathbf{R}_+)$ 为严格递增函数,$w(0) = 0, \lim_{t\to\infty} w(t) = \infty$;

(A6) $u \in C([-h,T),\mathbf{R}_+)$.

定理 5 如果 $u(t)$ 满足不等式(3)以及以上 6 个条件,从(A1)—(A6).那么不等式(3)的解是:

$$u(t) \leqslant w^{-1}\left[W^{-1}\left(W(g(t)) + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(t,s)\mathrm{d}s\right)\right], \quad t \in [t_0,T).$$

其中

$$W(r) = \int_{r_0}^{r}\frac{1}{H^m(w^{-1}(s))G^m(w^{-1}(s))}\mathrm{d}s, \quad 0 \leqslant r < \infty,$$

r_0 是一个合适的非负常数,使得 $W(r)$ 有定义.

$$H(r) = \max_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}\{H_{ij}(r)\},$$

$$G(r) = \max_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}\{G_{ij}(r)\}.$$

证明 取 η 为一任意常数满足 $t_0 \leqslant t \leqslant \eta < T$.根据条件(A1)和(A3),不等式(3)可写为:对 $t \in [t_0,\eta]$,

$$w(u(t)) \leqslant g(\eta) + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(\eta,s)\prod_{j=1}^{m} H_{ij}(u(s))G_{ij}\max_{s-h\leqslant\xi\leqslant s} u(\xi)\Big)\mathrm{d}s.$$

对 $t_0 \leqslant t \leqslant \eta$,定义上面的不等式的右边为 $Z(t)$,即

$$Z(t) = g(\eta) + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(\eta,s)\prod_{j=1}^{m} H_{ij}(u(s))G_{ij}(\max_{s-h\leqslant\xi\leqslant s} u(\xi))\mathrm{d}s.$$

不难看出,$Z(t)$ 非减,且 $0 \leqslant w(u(t)) \leqslant Z(t)$,$t \in [t_0,\eta]$.

由(A5),w^{-1} 存在且具有和 w 相同的性质.因此,

$$u(t) \leqslant w^{-1}(Z(t)), \quad t_0 \leqslant t \leqslant \eta.$$

此外,

$$\max_{s-h\leqslant\xi\leqslant s} u(\xi) \leqslant \max_{s-h\leqslant\xi\leqslant s} w^{-1}(Z(\xi)) = w^{-1}(Z(s)), \quad t_0 \leqslant s \leqslant \eta.$$

因此,对 $t \in [t_0,\eta]$,

$$Z(t) \leqslant g(\eta) + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(\eta,s)\prod_{j=1}^{m} H_{ij}(w^{-1}(Z(s)))G_{ij}(w^{-1}(Z(s)))\mathrm{d}s \leqslant$$

$$g(\eta) + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(\eta,s)\prod_{j=1}^{m}$$

$$H(w^{-1}(Z(s)))G(w^{-1}(Z(s)))\mathrm{d}s \leqslant g(\eta) + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(\eta,s) H^m(w^{-1}(Z(s)))G^m(w^{-1}(Z(s)))\mathrm{d}s.$$

再定义上面的不等式的右边为 $R(t)$,即:

$$R(t)=g(\eta)+\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(\eta,s)H^m(w^{-1}(Z(s)))G^m(w^{-1}(Z(s)))\mathrm{d}s,\quad t_0\leqslant t\leqslant \eta < T.$$

显然,$R(t_0)= g(\eta)$,$0\leqslant Z(t)\leqslant R(t)$,$t_0\leqslant t\leqslant \eta < T$.

$$\begin{aligned}R'(t)&=\sum_{i=1}^{n} f_i(\eta,\alpha_i(t))H^m(w^{-1}(Z(\alpha_i(t))))\cdot G^m(w^{-1}(Z(\alpha_i(t))))\alpha_i{}'(t)\leqslant\\&\sum_{i=1}^{n} f_i(\eta,\alpha_i(t))H^m(w^{-1}(Z(t)))\cdot G^m(w^{-1}(Z(t)))\alpha_i{}'(t)\leqslant\\&\sum_{i=1}^{n} f_i(\eta,\alpha_i(t))H^m(w^{-1}(R(t)))\cdot G^m(w^{-1}(R(t)))\alpha_i{}'(t)\leqslant\\&H^m(w^{-1}(R(t)))G^m(w^{-1}(R(t)))\sum_{i=1}^{n} f_i(\eta,\alpha_i(t))\alpha_i{}'(t).\end{aligned}$$

所以,

$$\frac{R'(t)}{H^m(w^{-1}(R(t)))G^m(w^{-1}(R(t)))}\leqslant \sum_{i=1}^{n} f_i(\eta,\alpha_i(t))\alpha_i{}'(t).$$

即:

$$\frac{\mathrm{d}(W(R(t)))}{\mathrm{d}t}\leqslant \sum_{i=1}^{n} f_i(\eta,\alpha_i(t))\alpha_i{}'(t).$$

对上式从 t_0 到 t 积分,$t\in[t_0,\eta]$,我们得到:

$$W(R(t))-W(R(t_0))\leqslant \int_{t_0}^{t}\sum_{i=1}^{n} f_i(\eta,\alpha_i(s))\alpha_i{}'(s)\mathrm{d}s=\int_{\alpha(t_0)}^{\alpha(t)}\sum_{i=1}^{n} f_i(\eta,s)\mathrm{d}s,$$

$$W(R(t))\leqslant W(g(\eta))+\sum_{i=1}^{n}\int_{\alpha(t_0)}^{\alpha(t)} f_i(\eta,s)\mathrm{d}s.$$

因为 W 连续且严格增,所以 W^{-1} 存在、连续、严格增.对上式使用 W^{-1},我们得到:

$$R(t)\leqslant W^{-1}\left(W(g(\eta))+\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(\eta,s)\mathrm{d}s\right),\quad t\in[t_0,\eta],$$

$$w(u(t))\leqslant Z(t)\leqslant R(t)\leqslant W^{-1}\left(W(g(\eta))+\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(\eta,s)\mathrm{d}s\right),\quad t\in[t_0,\eta],$$

$$u(t)\leqslant w^{-1}\left[W^{-1}\left(W(g(\eta))+\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(\eta,s)\mathrm{d}s\right)\right],\quad t\in[t_0,\eta].$$

在上式,取 $t=\eta$,可得:

$$u(\eta)\leqslant w^{-1}\left[W^{-1}\left(W(g(\eta))+\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(\eta)} f_i(\eta,s)\mathrm{d}s\right)\right],\quad t\in[t_0,\eta].$$

由于 η 是区间 $t_0\leqslant\eta < T$,内的一个任意数,我们得到:

$$u(t)\leqslant w^{-1}\left[W^{-1}\left(W(g(t))+\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(t,s)\mathrm{d}s\right)\right],\quad t\in[t_0,T).$$

证毕.

2 应用

我们的结果可以很容易地应用到微分方程的稳定性理论中.例如,我们考虑滞后型泛函微分方程

$$\frac{\mathrm{d}u}{\mathrm{d}t}=F(t,u_t),\tag{4}$$

在用 Lyapunov 第二方法研究微分方程稳定性时,如下条件经常可见:

1) $W_1(|\phi(0)|)\leqslant V(t,\phi) < W_2(\|\phi\|)$;

2) $W_1(|\phi(0)|)\leqslant V(t,\phi) < W_2(|\phi(0)|)+W_3\left(\int_{-h}^{0}|\phi(s)|^2\mathrm{d}s\right)$;

3) $W_1(|u(t)|_X)\leqslant V(t,u_t)\leqslant W_2(D(t,u_t))+\int_{t-h}^{t}L(s)W_1(|u(s)|_X)\mathrm{d}s$;

4) $V'_{(4)}(t,\phi)\leqslant 0$;

5) $V'_{(4)}(t,\phi)\leqslant -W_4(|\phi(0)|)$;

6) $V'_{(4)}(t,\phi)\leqslant -W_4(\int_{t-h}^{t}|\phi(s)|\mathrm{d}s)$;

7) $V_{(4)}(t,u_t)\leqslant -\eta(t)W_2(D(t,u_t))+P(t)$.

关于与这些条件相关的假设,定理可在众多的文献中查到,如文献[4-5,12-20].这些条件加上定理5,可以得到对 Lyapunov 函数和解的上界,文献[13]有一个很相近的例子.再如,定理 5 可以很容易地应用到 Volterra 积分方程:

$$X(t)=a(t)+\int_{t-\alpha}^{t}g(t,s)X(s)\mathrm{d}s,\quad X\in\mathbf{R}^n.\tag{5}$$

关于方程(5),很多论文研究过,如文献[5,21-22].

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A nonlinear Volterra-type integral inequality

WANG Tingxiu[1]

1 Department of Mathematics, Texas A&M University-Commerce, Commerce, TX 75428 USA

Abstract In this paper, we briefly review the development of research on Gronwall's inequality, then obtain a result for the following nonlinear Volterra-type integral inequality:

$$w(u(t)) \leqslant g(t) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(t,s) \prod_{j=1}^{m} H_{ij}(u(s)) G_{ij}(\max_{s-h \leqslant \xi \leqslant s} u(\xi))\, \mathrm{d}s.$$

Key words nonlinear integral inequalities; Gronwall inequality; Gronwall-type inequality; Volterra integro-differential inequalities; Lyapunov's second method