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随机镇定与反镇定概述

摘要

本文概述了随机镇定与反镇定理论的研究现状.主要回顾了一个微分方程的随机镇定与反镇定普遍理论及其发展,并围绕该理论的应用和扩展从四个方面阐述连续时间系统噪声镇定理论的当前发展概况.此外,本文还概述了离散时间系统随机镇定方面的最新进展.

关键词

几乎必然稳定性;连续时间系统;离散时间系统;噪声镇定;随机微分方程;随机镇定

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0 导读

本文原文为英文,希望感兴趣的读者进一步关注原文.

实际系统中通常存在噪声干扰,而在很多案例中这些噪声干扰不利于系统的运作,在较大程度上损害系统的良好性态,甚至破坏了系统的稳定性.长期以来,怎么处理系统运作中的噪声干扰是工程理论研究中的一个重要课题(参见文献[1-6]).

不但有如何克服噪声干扰保持稳定性的工作,而且不少研究发现利用噪声既可使系统失去稳定性却亦可能使不稳定的系统镇定或稳定的系统更加稳定.后者引起了人们广泛的研究兴趣,由此得到了许多重要的研究成果(见文献[5,7-8,10-12,15-24,26,29,33,36-37,42,68,72-75]等).利用噪声使(不稳定)系统镇定是有重要意义的研究课题,非常有助于工程系统的分析和设计.

在这些重要结果中,本文主要概述文献[20]提出的微分方程的随机镇定与反镇定普遍理论及其发展[15,21],并以这一理论的应用和扩展从以下四个方面阐述当前连续时间系统噪声镇定理论的发展概况(每一方面均列举例子并配以一详细范例说明):1)随机镇定理论的应用:如文献[26]应用文献[20](Theorem 3.1,亦见文献[5])中的随机镇定理论结合矩阵不等式提出利用噪声镇定的状态反馈控制器设计方法;2)控制策略及方式扩展至随机镇定理论:如文献[36]将采样数据控制策略(参见文献[61-63])扩展至随机镇定理论(参见文献[20],Theorem 3.1);3)随机镇定理论推广至多类系统:如文献[37]将对微分方程的随机镇定理论[20]推广至具有马氏切换的混合微分方程;4)利用其他类型噪声:如文献[73]以 Lévy 噪声替换文献[20]结果中的以布朗运动描述的噪声,得到了文献[20](Theorem 3.1)的一个推广结果.

相对于连续时间系统随机镇定理论及应用,离散时间系统的随机镇定理论与应用仍然处于初级阶段.在离散时间系统分析和设计中,噪声通常作干扰处理.在离散时间系统随机镇定方面,本文主要概述了文献[28]提出的新理论及其应用在状态反馈控制器设计上的新方法.

随机镇定是一个有重要意义而且内容丰富的研究领域,其中很多问题有待研究:例如上述噪声镇定理论的发展和应用主要针对具有线性增长条件的微分方程,对(高阶)非线性的微分方程也可作相应发展;控制策略和方式拓展在随机镇定上的问题值得进一步探究(文献[31,36]);离散时间系统噪声镇定理论与应用有许多需要研究的问题等.

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Stochastic stabilization and destabilization: A survey

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Abstract This paper provides an overview of recent advances of theory on stochastic stabilization and destabilization. The paper reviews a general theory on stochastic stabilization and destabilization of continuous-time systems and then illustrates the developments of theory on stabilization by noise with the applications and generalizations of this theory from four aspects. Moreover, this paper reviews some recent development of stochastic stabilization for discrete-time systems.

Key words almost sure stability; continuous-time systems; discrete-time systems; stabilization by noise; stochastic differential equations; stochastic stabilization

0 Introduction

A real-world system is usually subject to noise/disturbance, which could be the cause of poor performance or even instability of the system. For a long time, noise has been a challenging problem that has to be dealt with in engineering (see [1-6] among others). Traditionally, noise is treated as disturbance in the engineering literature (see, e.g., [1-2,6] and the references therein) and, therefore, an effective way for hanling it is to make the systems robust so that the controlled systems can counteract the effect of noise to keep good performance and/or stability. One can easily find the huge amounts of literature on this research. But people also found a phenomenum that a system, which could be unstable, is made stable by noise in several different contexts (see [7-14]).

It is now well known that noise can not only be used to destabilize a given stable system but also be used to stabilize a given unstable system or to make a system even more stable. The literature on stabilization and destabilization by noise is extensive (see [7, 10-12, 15-24] and the reference therein). Stabilization by deterministic periodic "noise", e.g., deterministic vibrational control, was investigated a great deal in the references (see, e.g., [9, 11-12, 14, 25]. But the study in the case of random noise, viz, stochastic stabilization, was initiated by Has'minskii who employed two white noise sources to stabilize a differential equation[10]. Later, the problems of stabilization and destabilization by random noise were studied in many works. This paper reviews the current development of theory on stochastic stabilization and destabilization. Especially interesting are the cases when the deterministic system is unstable while, after adding (multiplicative) noise, the corresponding stochastic system is stable (i.e. an unstable system is made stable by noise). This is very helpful in analysis and design of control systems (see, e.g., [26-28]). So this review will focus more on the development of theory for stabilization by noise.

The rest of the paper is organized as follows. Section 1 reviews a general theory on stochastic stabilization and destabilization proposed in [20] and then developed in [15] and [21]. In Section 2, the applications and developments of theory on stabilization by noise is reviewed from the following aspects (with a specified example as well as other examples for each): 1) applications of the theory, 2) extension of control strategies/methods/notions to stochastic stabilization, 3) generalizations to many other systems, and 4) using different types of noise. Section 3 reviews some novel results for stochastic stabilization of discrete-time systems proposed in [28]. Conclusions are given in Section 4.

1 A general theory on stochastic stabilization and destabilization

As is well known, noise can not only be used to destabilize a given stable system but also be used to stabilize a given unstable system or to make a system even more stable. Since the study in the case of random noise, viz, stochastic stabilization, initiated by Has'minskii who employed two white noise sources to stabilize a differential equation[10], the problems of stabilization and destabilization by random noise have been studied in many works, see, e.g., [7-8, 15, 17, 20-24, 26, 29-36]. Among these results, Arnold et al.[7] showed, in particular, that the system $\dot{x}(t)=Ax(t)$ can be stabilized by zero-mean stationary parameter noise if and only if the trace of matrix a is negative, i.e., *trace* $(A)<0$. And then the behavior of Lyapunov exponent of linear stochastic systems was studied in [22-23]. Especially interesting are the cases when the top Lyapunov exponent of the deterministic system is greater than zero while the top Lyapunov exponent of the corresponding stochastic system is smaller than zero (i.e. an unstable system is made stable by noise). On the other hand, Scheutzow[24] provided some examples on stabilization and destabilization in the plane, and Mao[20] developed a general theory on stabilization and destabilization by Brownian motion. But the conditions in [20] excluded many nonlinear systems. Appleby et al.[15] lifted the restrictions and extended the theory to a general class of nonlinear systems. Recently, Huang[21] further developed the theory and revealed more fundamental principles for stochastic stabilization and destabilization.

According to this general theory on stochastic stabilization and destabilization[15,20-21], a nonlinear ordinary differential equation (ODE)

$$\dot{x}=f(x(t),t) \quad t>0;\ x(0)=x_0\in\mathbf{R}^n\setminus\{0\} \qquad (1)$$

could be stabilized (resp. destabilized) using Brownian motion. More precisely, one could find some appropriate function $:\mathbf{R}^n\times\mathbf{R}_+\to\mathbf{R}^{n\times m}$ such that the equilibrium solution of stochastic differential equation (SDE)

$$\mathrm{d}x(t)=f(x(t))\mathrm{d}t+g(x(t))\mathrm{d}w(t) \quad t>0;$$
$$x(0)=x_0\in\mathbf{R}^n\setminus\{0\} \qquad (2)$$

is almost surely (a.s.) stable (resp. unstable), where $w(t)=[w_1(t) \ \cdots \ w_m(t)]^{\mathrm{T}}, t\geqslant 0$, is an m-dimensional Brownian motion; both $f:\mathbf{R}^n\times\mathbf{R}_+\to\mathbf{R}^n$ and $g:\mathbf{R}^n\times\mathbf{R}_+\to\mathbf{R}^{n\times m}$ obey

$$f(0,t)=0 \quad \text{and} \quad g(0,t)=0 \qquad (3)$$

for all $t\geqslant 0$ and satisfy the local Lipschitz continuous condition, that is, for any integer $k\geqslant 1$, there is $L_k>0$ such that

$$|f(x,t)-f(y,t)|\vee|g(x)-g(y)|\leqslant L_k(|x-y|) \qquad (4)$$

for all $(x,y)\in\mathbf{R}^n\times\mathbf{R}^n$ with $|x|\vee|y|\leqslant k$ and $t\geqslant 0$. Obviously, both systems (1) and (2) admit the equilibrium solution $x(t)=0$ for all $t\geqslant 0$ when initial condition $x_0=0$. Therefore, the noise perturbation preserves the equilibrium of the system (1).

By virtue of the local Lipschitz continuous condition (4), there is a unique continuous adapted process x (see, e.g., [5,37]) such that

$$x(t\wedge\tau_k)=x_0+\int_0^{t\wedge\tau_k}f(x(s))\mathrm{d}s+$$
$$\int_0^{t\wedge\tau_k}g(x(s))\mathrm{d}w(s), \quad t\geqslant 0 \quad \text{a.s.} \qquad (5)$$

where $\tau_k=\inf\{t>0:|x(t;x_0)|\geqslant k\}$. Here set $\inf\varnothing=\infty$ as usual. The equation (2) has a global solution if the explosion time τ_e defined by

$$\tau_e=\tau_e^{x_0}=\lim_{k\to\infty}\tau_k=$$
$$\inf\{t>0:|x(t;x_0)|\notin[0,\infty)\}. \qquad (6)$$

obeys $\tau_e=\infty$ a.s. In this work, it is important to show that solutions to equation (2) cannot reach zero in finite time, i.e., $\mathrm{P}(\{\theta_0<\infty\})=0$, where the stopping time θ_0 is defined by

$$\theta_0=\theta_0^{x_0}=\inf\{t>0:|x(t;x_0)|=0\}. \qquad (7)$$

The existing results (see, e.g., [15] and [5]) give $\tau_e\leqslant\theta_0$ and $\theta_0=\infty$ a.s. For the existence and uniqueness of global solutions of SDE (2), one has [21] (Proposition 3.1)

Proposition 1 Suppose that there is a function $\lambda:\mathbf{R}^n\to\mathbf{R}$ such that

$$\frac{1}{|x|^4}[|x|^2(2x^{\mathrm{T}}f(x,t)+$$
$$|g(x,t)|_F^2)-2|x^{\mathrm{T}}g(x,t)|^2]\leqslant\lambda(x). \qquad (8)$$

for all $x\neq 0$ and $t\geqslant 0$, where $|\cdot|$ is the Euclidean norm of a vector and its induced norm of a matrix; $|\cdot|_F$ is the Frobenius norm of a matrix; $\lambda(\cdot)$ is an upper-bounded continuou function, i.e., there is a constant H_λ such that $\lambda(x)\leqslant H_\lambda$ for all $x\in\mathbf{R}^n$. Then there exists a unique continuous adapted process x, which is a solution of (2) such that $\tau_e=\theta_0=\infty$ a.s.

Letting $\lambda(x)\equiv 0$ for all $x\in\mathbf{R}^n$ leads to [15] (Proposition 7).

1.1 Stochastic stabilization of continuous-time systems

In [20], it is assumed that function $f:\mathbf{R}^n\times\mathbf{R}_+\to\mathbf{R}^n$ satisfies a global linear bound

$$|f(x,t)|\leqslant K|x|,\quad \forall x\in\mathbf{R}^n, t\geqslant 0,\quad (9)$$

for some $K>0$, and function $g:\mathbf{R}^n\times\mathbf{R}_+\to\mathbf{R}^{n\times m}$ is chosen to be of the form

$$g(x,t)=[B_1\ \cdots\ B_m]x=[B_1x\ \cdots\ B_mx],\quad (10)$$

for all $t\geqslant 0$, where B_k, $1\leqslant k\leqslant m$, are all $n\times n$ constant matrices. So the stabilizing (resp. destabilizing) noise is of the form

$$g(x,t)\mathrm{d}w(t)=[B_1x\ \cdots\ B_mx][w_1(t)\ \cdots\ w_m(t)]^{\mathrm{T}}=\sum_{k=1}^m B_kxw_k(t).\quad (11)$$

and hence SDE (2) becomes

$$\mathrm{d}x(t)=f(x(t),t)\mathrm{d}t+\sum_{k=1}^m B_kx(t)\mathrm{d}w_k(t),\quad t>0;\ x(0)=x_0\in\mathbf{R}^n\backslash\{0\}.\quad (12)$$

For SDE (12), one has

Theorem 1 (see [20] Theorem 3.1) Let condition (9) hold and $\lambda>0$, $\rho\geqslant 0$. Assume

$$\sum_{k=1}^m|B_kx|^2\leqslant\lambda|x|^2\quad\text{and}\quad\sum_{k=1}^m|x^{\mathrm{T}}B_kx|^2\geqslant\rho|x|^4,\quad (13)$$

for all $x\in\mathbf{R}^n$. Then

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)|\leqslant-\left(\rho-K-\frac{1}{2}\lambda\right)\quad\text{a.s.}\quad (14)$$

for any $x_0\neq 0$. In particular, if $\rho>K+\frac{1}{2}\lambda$, then SDE (12) is almost surely exponentially stable.

Clearly, the global linear bound condition (9) is restrictive. Although, under some conditions, (9) can be weakened to a one-sided growth condition (see, e.g., [37] and [38]), many dynamical systems with equilibria at 0 are still excluded[15]. Appleby et al.[15] lifted the restriction and extended the theory on stochastic stabilization and destabilization to the general class of nonlinear systems (2), which satisfies conditions (3) and (4).

Theorem 2 (see [15] Theorem 8) Suppose that there is $\alpha\in(0,1)$ such that

$$|x|^2(2x^{\mathrm{T}}f(x,t)+|g(x,t)|_F^2)-(2-\alpha)|x^{\mathrm{T}}g(x,t)|^2\leqslant 0,\quad (15)$$

for all $t\geqslant 0$ and

$$\underline{g}(L):=\min_{|x|=L}|x^{\mathrm{T}}g(x,t)|>0,\quad (16)$$

for every $L>0$ and $t\geqslant 0$. Then, there exists a unique continuous adapted process x, which is a global solution of SDE (2) and which obeys

$$\lim_{t\to\infty}x(t)=0\quad\text{a.s.}$$

To reveal the more fundamental principles for stochastic stabilization and destabilization, Huang[21] further developed the theory for the general class of nonlinear systems (2).

Theorem 3 (see [21] Theorem 3.1) Suppose that there are functions $\mu\in C_\kappa(\mathbf{R}^n,\mathbf{R}_+;\mathbf{R}_+)$ and $\nu\in C(\mathbf{R}^n,\mathbf{R}_+;n\mathbf{R}_+)$ such that

$$\frac{1}{|x|^4}[|x|^2(2x^{\mathrm{T}}f(x,t)+|g(x,t)|_F^2)-2|x^{\mathrm{T}}g(x,t)|^2]\leqslant-\mu(x),\quad (17)$$

$$\frac{|x^{\mathrm{T}}g(x,t)|^2}{|x|^4}\leqslant\nu(x),\quad (18)$$

for all $t\geqslant 0$ and

$$\inf_{|x|\geqslant\beta}\frac{\mu(x)}{\nu(x)}>0\quad (19)$$

for every $\beta>0$, where $C(\mathbf{R}^n,\mathbf{R}_+;\mathbf{R}_+)$ is the family of nonnegative continuous functions $\mu:\mathbf{R}^n\times\mathbf{R}_+\to\mathbf{R}_+$ and $C_\kappa(\mathbf{R}^n,\mathbf{R}_+;\mathbf{R}_+)$ the family of nonnegative continuous functions $\mu:\mathbf{R}^n\times\mathbf{R}_+\to\mathbf{R}_+$ such that $\mu(x)>0$ for all $x\neq 0$, $0\leqslant\mu(0)<\infty$, and, moreover, $\liminf_{|x|\to\infty}\mu(x)=\infty$ if $\mu(0)=0$. Then there exists a unique continuous adapted process x, which is a global solution of SDE (2) and which obeys

$$\lim_{t\to\infty}x(t)=0\quad\text{a.s.}$$

The conditions (17)—(19) in Theorem 3 exploit the system structure and yield a more general result. It can be observed that Theorem 2 is a special case of Theorem 3 when $\mu(x)/\nu(x)>c>0$ for all $x\in\mathbf{R}^n$ while Theorem 1 can be considered as a specialized version with $\mu(x)=a>0$ and $b_2\geqslant\nu(x)\geqslant b_1>0$ for all $x\in\mathbf{R}^n$.

1.2 Stochastic destabilization of continuous-time systems

Let us now turn to consider the problems of stochastic destabilization, that is, a system, even if it is stable, can be made unstable by noise, which has been addressed in many works[1-2,6]. Particularly, Mao[20] proposed a theory on how to destabilize an ODE using multiplicative noise described with Brownian motion. It should be noticed that the multiplicative noise cannot destabilize a scalar ODE but make it even more stable if the ODE is a stable system (see, e.g., [5, 37]). Let $n \geqslant 2$ for the problems of stochastic destabilization.

In [20], an ODE that obeys the global linear bound condition (11) can be destabilized by multiplicative noise (11) for an appropriate choice of the set of matrices $B_k, 1 \leqslant k \leqslant m$.

Theorem 4 (see [20] Theorem 4.1) Let condition (9) hold and $\lambda > 0, \rho \geqslant 0$. Assume

$$\sum_{k=1}^{m} |B_k x|^2 \geqslant \lambda |x|^2 \quad \text{and}$$

$$\sum_{k=1}^{m} |x^{\mathrm{T}} B_k x|^2 \leqslant \rho |x|^4 \tag{20}$$

for all $x \in \mathbf{R}^n$. Then

$$\liminf_{t \to \infty} \frac{1}{t} \log |x(t)| \geqslant \left(\frac{1}{2} \lambda - K - \rho \right) \quad \text{a.s.} \tag{21}$$

for any $x_0 \neq 0$. In particular, if $\lambda > 2(K + \rho)$, then the solution $|x(t)|$ of SDE (12) tends to infinity almost surely exponentially fast, i.e., SDE (12) is almost surely exponentially unstable.

The theory was improved and extended to the general class of nonlinear systems (2) in Appleby et al.[15] by lifting the restrive global linear bound condition (9).

Theorem 5 (see [15] Theorem 12) Suppose that there is $\alpha \in (0,1)$ such that

$$|x|^2 (2x^{\mathrm{T}} f(x,t) + |g(x,t)|_F^2) - (2 + \alpha) |x^{\mathrm{T}} g(x,t)|^2 \geqslant 0. \tag{22}$$

for all $t \geqslant 0$. Let x be the unique continuous adapted process which is a solution of SDE (2) on $[0, \tau_e)$. Then, x obeys

$$\liminf_{t \to \tau_e} |x(t)| > 0 \qquad \text{a.s.}$$

Recently, Huang[21] further developed the theory and revealed the more fundamental principle for stochastic stabilization.

Theorem 6 Suppose there exist functions $\mu \in C_\kappa(\mathbf{R}^n, \mathbf{R}_+; \mathbf{R}_+)$ and $\nu \in C(\mathbf{R}^n, \mathbf{R}_+; \mathbf{R}_+)$ such that

$$\frac{1}{|x|^4} [|x|^2 (2x^{\mathrm{T}} f(x,t) + |g(x,t)|_F^2) - 2|x^{\mathrm{T}} g(x,t)|^2] \geqslant \mu(x), \tag{23}$$

$$\frac{|x^{\mathrm{T}} g(x,t)|^2}{|x|^4} \leqslant \nu(x), \tag{24}$$

for all $t \geqslant 0$ and

$$\inf_{|x| \leqslant \beta} \frac{\mu(x)}{\nu(x)} > 0, \tag{25}$$

for every $\beta > 0$. Let x be the unique continuous adapted process which is a solution of SDE (2) on $[0, \tau_e)$, where τ_e is defined by (6). Then, x obeys

$$\limsup_{t \to \tau_e} |x(t; x_0)| = \infty \qquad \text{a.s.}$$

Again, conditions (23)—(25) in Theorem 6 give more general and applicable results by exploiting the system structure. As is observed, Theorem 5 is a speical case of Theorem 4 when $\mu(x)/\nu(x) > c > 0$ for all $x \in \mathbf{R}^n$ while Theorem 6 can be considered as a version of Theorem 6 with $\mu(x) = a > 0$ and $b_2 \geqslant \nu(x) \geqslant b_1 > 0$ for all $x \in \mathbf{R}^n$. Moreover, Theorem 6 shows that, under these conditions, the solution $x(t)$ of SDE (2) is almost surely unbounded, which is desired for the purpose of stochastic destabilization.

2 Development of theory for stabilization by noise

Over the past few decades, stochastic systems have been intensively studied since stochastic modelling has come to play an important role in science and engineering (see, e.g., [4-5, 28, 39-46]). On one hand, many results for stochastic systems have found applications. The general theory reviewed above has been applied to study feedback stabilization problems of stochastic systems, where the results on mean-square stability may be not applicable. For instance, [27] applied the general theory on stochastic stabilization[15,20-21] and presented a noise-assisted stabilization method for nonlinear systems based on the notion of control Lyapunov functions (see, e.g., [41, 44]); Hu and Mao[26] applied a result for stochastic stabilization (see Theorem 1 above), [20] (Theorem 3.1) and also [5]) and used the techniques of linear matrix inequalities (LMIs)[47] to study almost sure stabilization of linear stochastic systems

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\mathrm{d}t + \sum_{i=1}^{m} [C_i x(t) + D_i u(t)]\mathrm{d}w_i(t), \tag{26}$$

where one needs to design a state-feedback controller $u(t)=Kx(t)$ such that the closed-loop system

$$\mathrm{d}x(t)=[A+BK]x(t)\mathrm{d}t+\sum_{i=1}^{m}[C_i+D_iK]x(t)\mathrm{d}w_i(t) \quad (27)$$

is almost surely stable.

Theorem 7 (see [26] Theorem 2) The equilibrium of the stochastic system (27) is almost surely exponentially stable with respect to state-feedback gain $K=YX^{-1}$, if there exists a positive definite matrix X, a matrix Y and real numbers $\alpha_i \geqslant 0$ $(1 \leqslant i \leqslant m)$ such that following LMIs hold:

$$\begin{bmatrix} \Pi_{11}-\alpha X & \Pi_{21}^{\mathrm{T}} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} < 0, \quad (28)$$

and, for each $i=1,2,\cdots,m$, either

$$(C_iX+D_iY)^{\mathrm{T}}+(C_iX+D_iY)-\sqrt{2\alpha_i}X>0 \quad (29)$$

or

$$(C_iX+D_iY)^{\mathrm{T}}+(C_iX+D_iY)+\sqrt{2\alpha_i}X<0, \quad (30)$$

where $\Pi_{11}=(AX+BY)^{\mathrm{T}}+(AX+BY)$, $\Pi_{22}=\mathrm{diag}\{-X,-X,\cdots,-X\}$, $\Pi_{21}=[(C_1X+D_1Y)^{\mathrm{T}} \quad (C_2X+D_2Y)^{\mathrm{T}} \quad \cdots \quad (C_mX+D_mY)^{\mathrm{T}}]^{\mathrm{T}}$ and $\alpha=\sum_{i=1}^{m}\alpha_i$.

On the other hand, the theory of stochastic systems has been developed by extending control strategies/methods/notions to stochastic systems. For example, [3, 39-40] were dedicated to extension of H_∞-type control theory to stochastic systems; [48-51] developed sliding mode control (SMC) for stochastic systems; [52-54] presented Razumikhin-type theorems for stability/input-to-state stability of stochastic delay systems; some results studied stochastic stability/stabilization for switched systems (e.g., [55,57]); [58-59] studied the stabilization problems with input delay (in the drift) for stochastic systems while [31] investigated on stochastic stabilization with input delay (in the diffusion); recently, [60] extended the sampled data control schemes to stochastic systems; and, particularly, Mao[36] developed the theory of stochastic stabilization using the sampled data control strategy, which considered a stochastically controlled system by a scalar Brownian motion $B(t)$

$$\mathrm{d}x(t)=f(x(t))\mathrm{d}t+Ax([t/\tau]\,\tau)\mathrm{d}B(t),\quad t>0;\ x(0)=x_0 \in \mathbf{R}^n, \quad (31)$$

where $f:\mathbf{R}^n \to \mathbf{R}^n$, $A \in \mathbf{R}^{n\times n}$ and $[t/\tau]$ is the integer part of t/τ with $\tau>0$ being the sampling interval (see, e.g., [61-63]).

Theorem 8 (see [36] Theorem 3.3) Assume that f satisfies $f(0)=0$ and the globally Lipschitz continuous condition, that is,

$$|f(x)-f(y)|<\alpha|x-y|,\quad x,y \in \mathbf{R}^n, \quad (32)$$

for some $\alpha>0$, and assume that there are two positive constants ρ_1 and ρ_2 such that

$$\rho_2-\frac{1}{2}\rho_1>\alpha, \quad (33)$$

and, for all $x \in \mathbf{R}^n$,

$$|Ax|^2 \leqslant \rho_1|x|^2 \quad \text{and} \quad |x^{\mathrm{T}}Ax|^2 \geqslant \rho_2|x|^4. \quad (34)$$

Then there is a positive number τ^* such that, for any $x_0 \in \mathbf{R}^n$, the solution of Eq.(31) satisfies

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)|<0 \quad \text{a.s.}$$

provided $\tau \in (0,\tau^*)$. In practice, one can choose a pair of constant $p,\varepsilon \in (0,1)$ for [36] (Eq.(3.8)) to hold and let $\tau^*=\bar{\tau}$, where $\bar{\tau}$ is the unique root to [36] (Eq.(3.13)). This result may be thought as an extension of sampled data control methods to [20] (Theorem 3.1).

Moreover, the theory of stochastic stabilization has been generalized to many classes of systems (in addition to ODEs). For example, [64-65] extended vibrational stabilization to time-delay systems; [16-17, 33,66] extended the theory of stochastic stabilization to accomodate partial differential equations (PDEs); [29,67] were dedicated to extension of stochastic stabilization to time-delay systems described with functional/delay differential equations, where [21] suggested that the techniques in [15,21] can be applied to the class of nonliear functional differential equations in [67]; Mao et al.[37] generalized the results in [20] to hybrid systems of stochastic differential equations

$$\mathrm{d}x(t)=f(x(t),t,r(t))\mathrm{d}t+g(x(t),t,r(t))\mathrm{d}w(t),\quad t>0;\ x(0)=x_0 \in \mathbf{R}^n \quad (35)$$

where both $f:\mathbf{R}^n \times \mathbf{R}_+ \times S \to \mathbf{R}^n$ and $g:\mathbf{R}^n \times \mathbf{R}_+ \times S \to \mathbf{R}^{n\times m}$ satisfy the local Lipschitz condition with $f(0,t,i)=0$, $g(0,t,i)=0$ for all $t \geqslant 0$, $i \in S$, and grow most linearly with respect to x; process $r(t)$, $t\geqslant 0$, is a right-continuous Markov chain taking values in $S=\{1,2,\cdots,N\}$ with generator $\Gamma=(\gamma_{ij})_{N\times N}$ given by

$$\mathrm{P}\{r(t+\Delta)=j \mid r(t)=i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & i \neq j, \\ 1+\gamma_{ii}\Delta + o(\Delta), & i = j, \end{cases} \quad \Delta > 0$$

where $\gamma_{ij} \geqslant 0$ if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$; the Markov chain $r(t)$ has a unique stationary (probability) distribution $\pi = [\pi_1, \pi_2, \cdots, \pi_N]$ which can be determined by solving linear equation $\pi\Gamma = 0$ subject to $\sum_{j=1}^{N} \pi_j = 1$ and $\pi_j > 0$ for $j \in S$.

Theorem 9 (see [37] Theorem 3.3) Assume that, for each $i \in S$, there are constant triples α_i, ρ_i and σ_i such that

$$\begin{aligned} & x^{\mathrm{T}} f(x,t,i) \leqslant \alpha_i |x|^2, \\ & |g(x,t,i)|^2 \leqslant \rho |x|^2, \\ & |x^{\mathrm{T}} g(x,t,i)|^2 \geqslant \sigma_i |x|^2, \end{aligned} \tag{36}$$

for all $(x,t) \in \mathbf{R}^n \times \mathbf{R}_+$. Then the solution of Eq. (35) satisfies

$$\limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \leqslant \sum_{j=1}^{N} \pi_j \left(\alpha_j + \frac{1}{2}\rho_j^2 - \sigma_j^2\right) \quad \text{a.s.} \tag{37}$$

for all $x_0 \in \mathbf{R}^n$. In particular, the nonlinear hybrid SDE (35) is almost surely exponentially stable if

$$\sum_{j=1}^{N} \pi_j \left(\alpha_j + \frac{1}{2}\rho_j^2 - \sigma_j^2\right) < 0. \tag{38}$$

Also note that Huang and Mao[56] found that both the diffusion and the Markov chain play important roles in analysis and synthesis of singular systems and a singular system can be regularized and/or stabilized by noise.

Alternatively, the theory of stochastic stabilization has been generalized by using different types of noise. For example, [66] extended the stochastic stabilization of PDEs [16-17] by using Strotonovich noise; [68] studied stochastic stabilization of PDEs by Lévy noise; [69] extended the results on almost sure stability/stabilization of stochastic delay differential equations (see [48,70-71]) using Lévy noise; and, particularly, [72-73] gave the generalizations of some results in [5,20] using Lévy noise, which, e.g., examined the almost sure stability conditions for deterministic system (1) with global linear bound (9) that is perturbed by noise

$$\mathrm{d}x(t) = f(x(t-),t)\mathrm{d}t + \sum_{k=1}^{m} B_k x(t-)\mathrm{d}w_k(t) + \int_{|y|<r} D(y)x(t-)\widetilde{N}(\mathrm{d}t,\mathrm{d}y), \tag{39}$$

on $t \geqslant 0$, $x(0) = x_0 \in \mathbf{R}^n$, where $\widetilde{N}$ of the form $\widetilde{N} = N(\mathrm{d}t,\mathrm{d}y) - \nu(\mathrm{d}y)\mathrm{d}t$ with Lévy measure ν is the compensator of an $\mathcal{F}$-adapted Poisson random measure N, independent of Brownian motion $w(t)$, defined on $\mathbf{R}_+ \times (\mathbf{R}^m \setminus \{0\})$; positive number r is the maximum allowable jump size; $D: \mathbf{R}^m \to \mathbf{R}^{n \times n}$ is a measurable function. A generalization of [20] (Theorem 3.1) is given as follows.

Theorem 10 (see [73] Theorem 3.2) Assume that $\int_{|y|<r} (|D(y)| \wedge |D(y)|^2)\nu(\mathrm{d}y) < \infty$ and that $D(y)$ does not have any eigenvalue equal to -1 (ν almost everywhere). Moreover, assume that the following conditions are satisfied

$$\begin{aligned} & \sum_{k=1}^{m} |B_k x|^2 \leqslant \xi |x|^2, \\ & \sum_{k=1}^{m} |x^{\mathrm{T}} B_k x|^2 \geqslant \gamma |x|^4, \\ & \int_{|y|<r} x^{\mathrm{T}} D(y) x \nu(\mathrm{d}y) \geqslant \delta |x|^2, \end{aligned} \tag{40}$$

for all $x \in \mathbf{R}^n$, where $\xi > 0, \gamma \geqslant 0, \delta \geqslant 0$. Then the sample Lyapunov exponent of the solution of (39) exists and satisfies

$$\limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \leqslant -\left(\gamma - K - \frac{\xi}{2} - \int_{|y|<r} \log(1+|D(y)|)\nu(\mathrm{d}y) + \delta\right) \quad \text{a.s.} \tag{41}$$

for any $x_0 \neq 0$. If

$$\gamma > K + \frac{\xi}{2} - \delta + \int_{|y|<r} \log(1+|D(y)|)\nu(\mathrm{d}y)$$

then the trivial solution of the system (39) is almost surely exponentially stable.

3 Stochastic stabilization of discrete-time systems

As is well known, noise may play a stabilizing or destabilizing role in continuous-time systems. Almost sure stability and stabilization of stochastic differential equations, or say, continuous-time stochstic systems have received much attention. As is shown above, there are many results and they have been applied to control design methods for stabilization of continuous-time systems. However, relatively few works (see, e.g., [42,74-76]) are concerned with those problems of stochastic difference equations, or say, discrete-time stochastic systems. These existing works are mainly dedicated to almost sure stability of scalar systems, see [74-76] and the references therein. As is known, whenever a computer is used in measurement, computation, signal processing or control applications, the data, signals and systems involved are naturally described with discrete-time processes, see, e.g., [77-78]. Therefore, theory of discrete-time dynamic signals and systems is useful in design and analysis of control systems, signal filters, and state estimators from time-series of process data as well as scientific computations. As a matter of fact, discrete-time stochastic systems and those discretized from continuous-time stochastic systems have been intensively studied over the past few decades, see [42-43,45,74-75,79-80] and the references therein. Most of these results study the stochastic systems in mean-square sense since it is often relatively easier to analyze some properties such as stability and asymptotic behavior in this way[75]. But, as is well known, a mean-square unstable system could be almost surely asymptotically stable (or, simply, almost surely stable), which is concerned with all paths on the sample space and is desired in many practical cases. For analysis and design of discrete-time systems, noise was treated as disturbance in the literature until, most recently, Huang et al.[28] developed a theory for almost sure stability of discrete-time stochastic systems and, as an application, proposed a novel controller design method that exploits the stabilizing role of noise in discrete-time systems. The proposed theory employs a numerical result derived from the inspiring work of Higham[42]. More specifically, Huang et al.[28] defined a function $H_\varepsilon(\cdot)$ using a numerical result derived from the work [42] (see, e.g., [42] Fig.5.2), which is given in Figure 1 and plays an importan role in the proposed theory.

Let us consider a discrete-time stochastic system

$$x_{k+1}=f(x_k,k)+g(x_k,k)w_{k+1},\quad k\in\mathbf{Z}_+;\ x_0\in\mathbf{R}^n\setminus\{0\},\tag{42}$$

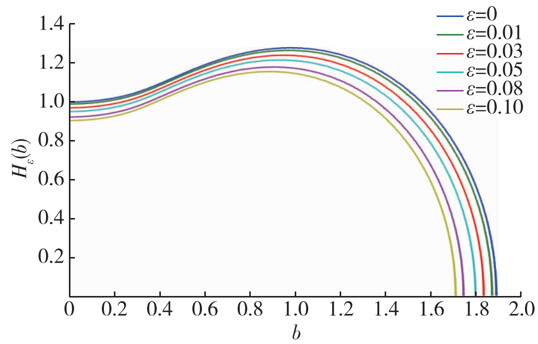

Fig. 1 Curves of function $a=H_\varepsilon(b)$ against b for given $\varepsilon\geqslant 0$

where functions $f:\mathbf{R}^n\times\mathbf{Z}_+\to\mathbf{R}^n$ and $g:\mathbf{R}^n\times\mathbf{Z}_+\to\mathbf{R}^n$ are such that, for each fixed $k\in\mathbf{Z}_+$, the functions $f(\cdot,k)$ and $g(\cdot,k)$ are both continuous mapping from $\mathbf{R}^n$ to $\mathbf{R}^n$; $\{w_k\}_{k\geqslant 1}$ is an independent and identically distributed (i.i.d.) sequence with w_k obeying Gaussian distribution $\mathcal{N}(0,1)$. Moreover, assume that

$$f(0,k)=g(0,k)=0,\qquad \forall\ k\in\mathbf{Z}_+,\tag{43}$$

and there is a function $\beta_L:\mathbf{R}_+\to\mathbf{R}_+$ with $\beta_L(|x|)>0$ for all $|x|>0$ such that

$$|f(x,k)|^2\vee|g(x,k)|^2\geqslant\beta_L(|x|),\tag{44}$$

for all $k\in\mathbf{Z}_+$.

For the general nonlinear discrete-time stochastic systems (42), Huang et al.[28] presented a criterion for almost sure stability that exploits the stabilizing role of noise in discrete-time systems.

Theorem 11 (see [28] Theorem 3.1) Assume that there exist real constants $\bar\lambda_f\geqslant 0$, $\bar\lambda_g\geqslant\underline\lambda_g>0$, sequence $\{\lambda_k\}$ with $\lambda_k>0$, and a positive definite matrix $P\in\mathbf{R}^{n\times n}$ such that, for all $k\geqslant 0$,

$$f^{\mathrm T}(x_k,k)Pf(x_k,k)\leqslant\bar\lambda_f^2x_k^{\mathrm T}Px_k,\tag{45}$$

$$\underline\lambda_g^2x_k^{\mathrm T}Px_k\leqslant g^{\mathrm T}(x_k,k)Pg(x_k,k)\leqslant\bar\lambda_g^2x_k^{\mathrm T}Px_k,\tag{46}$$

$$\pm[f^{\mathrm T}(x_k,k)Pg(x_k,k)+g^{\mathrm T}(x_k,k)Pf(x_k,k)]\geqslant \lambda_kf^{\mathrm T}(x_k,k)Pf(x_k,k)+\frac{1}{\lambda_k}g^{\mathrm T}(x_k,k)Pg(x_k,k)-\beta x_k^{\mathrm T}Px_k.\tag{47}$$

With $(\underline\lambda_g^2\wedge 1)\gg\beta\geqslant 0$. Then the solution of system (42) obeys

$$\limsup_{k\to\infty}\frac{1}{k}\log|x_k|<0\quad\text{a.s.}\tag{48}$$

provided there exists a number

$$2\varepsilon>\varepsilon_\beta^\alpha+\Delta(\underline\lambda_g,\varepsilon_\beta,\alpha)\tag{49}$$

for some $\alpha\in(0,1)$ such that

$$\bar{\lambda}_f < \min_{b\in[\underline{\lambda}_g,\bar{\lambda}_g]} H_\varepsilon(b), \tag{50}$$

Where $(\underline{\lambda}_g^2 \wedge 1) \gg \varepsilon_\beta = \dfrac{\bar{\lambda}_f}{\underline{\lambda}_g}\beta \geq 0$,

$$\Delta(\underline{\lambda}_g, \varepsilon_\beta, \alpha) = \int_0^{\varepsilon_\beta^{1-\alpha}/\underline{\lambda}_g^2} [(z/\underline{\lambda}_g^2) - 1 - \log z + \log \underline{\lambda}_g^2]\tilde{p}(z,1)\,dz,$$

and $\tilde{p}(z,\nu)$ is the probability density function of χ^2 random variable with ν degrees of freedom.

A specialized version of Theorem 11 for linear discrete-time stochastic system

$$x_{k+1} = Fx_k + Gx_k w_{k+1} \tag{51}$$

yields the following result.

Theorem 12(see[28]Theorem 4.1) Given constants $\lambda>0$, $\bar{\lambda}_F \geq 0$ and $\bar{\lambda}_G \geq \underline{\lambda}_G > 0$, assume that there exists a positive definite matrix $P \in \mathbf{R}^{n\times n}$ such that

$$F^{\mathrm{T}}PF \leq \bar{\lambda}_F^2 P, \tag{52}$$

$$\underline{\lambda}_G^2 P \leq G^{\mathrm{T}}PG \leq \bar{\lambda}_G^2 P, \tag{53}$$

$$\pm(F^{\mathrm{T}}PG + G^{\mathrm{T}}PF) \geq \lambda F^{\mathrm{T}}PF + \frac{1}{\lambda}G^{\mathrm{T}}PG - \beta P. \tag{54}$$

With $(\underline{\lambda}_G^2 \wedge 1) \gg \beta \geq 0$. If there exists $\varepsilon > 0$ satisfying (49) such that $\bar{\lambda}_F < \min_{b\in[\underline{\lambda}_G,\bar{\lambda}_G]} H_\varepsilon(b)$, then system (51) obeys

$$\limsup_{k\to\infty} \frac{1}{k}\log|x_k| < 0 \quad \text{a.s.}$$

For stochastic stabilization of linear discrete-time system

$$x_{k+1} = Fx_k,$$

one needs to find some matrix G such that stochastic system (51) is almost surely stable. It is noticed that Theorem 12 may not be directly applied to this problem since conditions (52)—(54) are no longer in the form of LMIs when G is a decision variable. But it is not difficult to derive the following result from Theorem 12 for the stochastic stabilization problem, where, by Schur complement lemma, conditions (44)—(47) in [28] (Theorem 4.2) can be easily rewritten in the form of LMIs.

Theorem 13(see[28]Theorem 4.2) Given constants $\lambda > 0$, $\bar{\lambda}_F \geq 0$ and $\bar{\lambda}_G \geq \underline{\lambda}_G > 0$, assume that there are matrices $P > 0$ and $\tilde{G}$ such that the following LMIs

$$F^{\mathrm{T}}PF \leq \bar{\lambda}_F^2 P, \tag{55}$$

$$\begin{bmatrix} -\bar{\lambda}_G^2 P & \tilde{G}^{\mathrm{T}} \\ \tilde{G} & -P \end{bmatrix} \leq 0, \tag{56}$$

$$\lambda F^{\mathrm{T}}PF + \frac{1}{\lambda}(\underline{\lambda}_G^2 P) \leq \pm(F^{\mathrm{T}}\tilde{G} + \tilde{G}^{\mathrm{T}}F), \tag{57}$$

$$\begin{bmatrix} \lambda F^{\mathrm{T}}PF - \beta P \mp (F^{\mathrm{T}}\tilde{G} + \tilde{G}^{\mathrm{T}}F) & \tilde{G}^{\mathrm{T}} \\ \tilde{G} & -\lambda P \end{bmatrix} \leq 0, \tag{58}$$

hold with $(\underline{\lambda}_G^2 \wedge 1) \gg \beta \geq 0$ and that there exists $\varepsilon>0$ satisfying (49) such that $\bar{\lambda}_F < \min_{b\in[\underline{\lambda}_G,\bar{\lambda}_G]} H_\varepsilon(b)$ Let $G = P^{-1}\tilde{G}$ then system (51) obeys

$$\limsup_{k\to\infty} \frac{1}{k}\log|x_k| < 0 \quad \text{a.s.}$$

As an application of the established results, [28] proposed a novel controller design method for almost sure stabilization of linear discrete-time stochastic system

$$x_{k+1} = Ax_k + Bu_k + Gx_k w_{k+1} \tag{59}$$

with matrix G of full rank, which requires to find a state-feedback controller $u_k = \tilde{K}x_k$ such that the closed-loop stochastic system (51) with $F = A + B\tilde{K}$ is almost surely stable. The following result is derived from Theorem 12 for the stabilization problem of (59).

Theorem 14 Given constants $\lambda>0$, $\bar{\lambda}_F \geq 0$ and $\bar{\lambda}_G \geq \underline{\lambda}_G \geq 0$, assume that there are matrices $X>0$ and $\tilde{Y}$ such that

$$\begin{bmatrix} -\dfrac{\bar{\lambda}_F^2}{\bar{\lambda}_G^2}X & * \\ \tilde{Z} & -X \end{bmatrix} \leq 0, \tag{60}$$

$$\begin{bmatrix} -\bar{\lambda}_G^2 X & * \\ GX & -X \end{bmatrix} \leq 0, \tag{61}$$

$$\begin{bmatrix} \mp(\tilde{Z}^{\mathrm{T}} + \tilde{Z}) & * & * \\ \tilde{Z} & -\dfrac{1}{\lambda}X & 0 \\ G^{-1}X & 0 & -\dfrac{\lambda}{\underline{\lambda}_G^2}X \end{bmatrix} \leq 0, \tag{62}$$

$$\begin{bmatrix} \mp(\tilde{Z}^{\mathrm{T}} + \tilde{Z}) + \dfrac{1}{\lambda}X - \beta_G X & * \\ \tilde{Z} & -\dfrac{1}{\lambda}X \end{bmatrix} \leq 0. \tag{63}$$

with $(\underline{\lambda}_G^2 \wedge 1) \gg \beta_G \geq 0$, where $\tilde{Z} = AG^{-1}X + B\tilde{Y}$ and entries denoted by * can be readily inferred from symmetry of a matrix, and assume that there exists $\varepsilon>0$ satisfying (49) with $\beta=\bar{\lambda}_G^2\beta_G$ such that $\bar{\lambda}_F<\min_{b\in[\lambda_G,\bar{\lambda}_G]}H_\varepsilon(b)$. Let the feedback gain matrix $\tilde{K}=\tilde{Y}X^{-1}G$, then the closed-loop system (59) with $u_k=\tilde{K}x_k$ obeys

$$\limsup_{k\to\infty}\frac{1}{k}\log|x_k|<0 \quad \text{a.s.}$$

The proposed control design method exploits the stabilizing role of noise in discrete-time systems and applies to some cases where the other results in the literature do not work, which have been verified with examples in [28].

4 Concluding remarks

This paper has given an overview of some recent advances of theory on stabilization and destabilization by noise. Particularly, this paper has reviewed a general theory proposed in [20] and developed in [15,21] as well as its applications and generalizations. It is observed, as reviewed in Section 3, that the results in [20] for systems with the linear growth condition have been used and/or generalized in many works (see, e.g., [26,36-37,73]) and now a few begin to apply the results developed in [15,21] for (highly) nonlinear systems (see, e.g., [27]). There is much work to do for (highly) nonlinear systems as well as developments of techniques such as input delay or sampled data control (in diffusion). This paper has also reviewed the theory on stochastic stabilization of discrete-time systems developed in [28]. It appears that, compared with that for continuous-time systems, the theory of stabilization by noise for discrete-time systems and its applications are in an early stage. In summary, the study of stochastic stabilization and destabilization is a very rich research field.

References

[1] Åström K J. Introduction to stochastic control theory[M]. New York, US: Dove Publications, 2006
[2] Borrie J A. Stochastic systems for engineers: Modelling, estimation and control[M]. New Jersey, US: Prentice Hall, 1992
[3] Gershon E, Shaked U. Static H_2 and H_∞ output-feedback of discrete-time LTI systems with state multiplicative noise[J]. Systems & Control Letters, 2006, 55(3): 232-239
[4] Huang L R. Stability and stabilisation of stochastic delay systems[D]. Glasgow, Scotland, UK: Department of Mathematics and statistics, University of Strathclyde, 2010
[5] Mao X R. Stochastic differential equations and applications[M]. 2nd edition. Chichester, UK: Horwood Publishing, 2007
[6] Willsky A S, Levy B C. Stochastic stability research for complex power systems[R]. DOE Contract, LIDS, MIT, Report ET-76-C-01-2295, 1979
[7] Arnold L, Crauel H, Wihstutz V. Stabilization of linear systems by noise[J]. SIAM J Control and Optimization, 1983, 21(3): 451-461
[8] Arnold L, Kloeden P. Lyapunov exponents and rotation number of two-dimensional systems with telegraphic noise [J]. SIAM J Appl Math, 1989, 49(4): 1242-1274
[9] Bellman R, Bentsman J, Meerkov S. Stability of fast periodic systems[J]. IEEE Trans Automatic Control, 1985, 30(3): 289-291
[10] Has'minskii R Z. Stochastic stability of differential equations[M]. Alphen aan den Rijn, Netherland: Sithoff & Noordhoff, 1980
[11] Meerkov S. Principle of vibrational control: Theory and applications[J]. IEEE Trans Automatic Control, 1980, 25(4): 755-762
[12] Meerkov S. Condition of vibrational stabilizability of a class of non-linear systems[J]. IEEE Trans Automatic Control, 1982, 27(2): 485-487
[13] Morrone J A, Markland T E, Ceriotti M, et al. Efficient multiple time scale molecular dynamics: using colored noise thermostats to stabilize resonances[J]. J Chem Phys, 2011, 134(1): 014103
[14] Zhabko A P, Kharitonov V L. Problem of vibrational stabilisation of linear systems[J]. Autom Telemekh, 1980 (2): 31-34
[15] Appleby J A D, Mao X R, Rodkina A. Stabilization and destabilization of nonlinear differenti equations by noise [J]. IEEE Trans Automatic Control, 2008, 53(3): 683-691
[16] Caraballo T, Liu K, Mao X R. On stabilization of partial differential equations by noise[J]. Nagoya Math J, 2001, 161: 155-170
[17] Caraballo T, Garrido-Atienza M, Real J. Stochastic stabilisation of differential systems with general decay rate [J]. Systems & Control Letters, 2003, 48(5): 397-406
[18] Cerrai S. Stabilization by noise for a class of stochastic reaction-diffusion equations[J]. Probab Theory Relat Fields, 2005, 133(2): 190-214
[19] Kushner H J. On the stability of processes defined by stochastic difference-differential equations[J]. J Differ Equ, 1968, 4(3): 424-443
[20] Mao X R. Stochastic stabilisation and destabilisation[J]. Systems & Control Letters, 1994, 23(4): 279-290
[21] Huang L R. Stochastic stabilization and destabilization of nonlinear differential equations[J]. Systems & Control Letters, 2013, 62(2): 163-169
[22] Pardoux E, Wihstutz V. Lyapunov exponent and rotation number of two-dimensional linear stochastic systems with

small diffusion[J]. SIAM J Appl Math, 1988, 48(2): 442-457
[23] Pardoux E, Wihstutz V. Lyapunov exponent of linear stochastic systems with large diffusion term[J]. Stoch Processes Appl, 1992, 40(2): 289-308
[24] Scheutzow M. Stabilization and destabilization by noise in the plane[J]. Stoch Anal Appl, 1993, 11(1): 97-113
[25] Bellman R, Bentsman J, Meerkov S. Vibrational control of nonlinear systems: Vibrational stabilizability[J]. IEEE Trans. Automatic Control, 1986, 31(8): 710-716
[26] Hu L J, Mao X R. Almost sure exponential stabilisation of stochastic systems by state-feedback control[J]. Automatica, 2008, 44(2): 465-471
[27] Hoshino K, Nishimura Y, Yamashita Y, et al. Global asymptotic stabilization of nonlinear deterministic systems using Wiener processes[J]. IEEE Trans Automatic Control, 2016, 61(8): 2318-2323
[28] Huang L R, Hjalmarsson H, Koeppl H. Almost sure stability and stabilization of discrete-time stochastic systems[J]. Systems & Control Letters, 2015, 82: 26-32
[29] Appleby J A D, Mao X R. Stochastic stabilisation of functional differential equations[J]. Systems & Control Letters, 2005, 54(11): 1069-1081
[30] Deng F Q, Luo Q, Mao X R, et al. Noise supresses or expresses exponential growth[J]. Systems & Control Letters, 2008, 57(3): 262-270
[31] Guo Q, Mao X R, Yue R X. Almost sure exponential stability of stochastic differential delay equations[J]. SIAM J Control and Optimization, 2008, 54(4): 1919-1933
[32] Khasminskii R Z, Zhu C, Yin G. Stability of regime-switching diffusions[J]. Stochastic Process Appl, 2007, 117(8): 1037-1051
[33] Kwiecińska A A. Stabilization of partial differential equations by noise[J]. Stoch Processes Appl, 1999, 79(2): 179-184
[34] Luo Q, Mao X R. Stochastic population dynamics under regime switching II[J]. J Math Anal Appl, 2009, 355(2): 577-593
[35] Mao X R, Marion G, Renshaw E. Environmental noise suppresses explosion in population dynamics[J]. Stoch Process Appl, 2002, 97(1): 95-110
[36] Mao X R. Almost sure exponential stabilization by discrete-time stochastic feedback control[J]. IEEE Trans Automatic Control, 2016, 61(6): 1619-1624
[37] Mao X R, Yin G G, Yuan C G. Stabilisation and destabilisation of hybrid systems of stochastic differential equations[J]. Automatica, 2007, 43(2): 264-273
[38] Wu F K, Hu S G. Suppression and stabilisation of noise[J]. Intern J Control, 2009, 82(11): 2150-2157
[39] Berman N, Shaked U. H_∞ like control for nonlinear stochastic systems[J]. Systems & Control Letters, 2006, 55(3): 247-257
[40] Berman N, Shaked U. H_∞ control for discrete-time nonlinear stochastic systems[J]. IEEE Trans. Automatic Control, 2006, 51(6): 1041-1046
[41] Deng H, Krstić M, Williams R J. Stabilization of stochastic nonlinear systems driven by noise of unknown covariance[J]. IEEE Trans Automatic Control, 2001, 46: 1237-1253
[42] Higham D J. Mean-square and asymptotic stability of the stochastic theta method[J]. SIAM J Numer Anal, 2001, 38(3): 753-769
[43] Higham D J, Mao X R, Yuan C G. Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations[J]. SIAM J Numer Anal, 2007, 45(2): 529-609
[44] Krstić M, Kokotović P V. Control Lyapunov functions for adaptive nonlinear stabilization[J]. Systems & Control Letters, 1995, 26(1): 7-23
[45] Saito Y, Mitsui T. Stability analysis of numerical schemes for stochastic differential equations[J]. SIAM J Numer Anal, 1996, 33(6): 2254-2267
[46] Song M H, Hu L J, Mao X R. Khasminskii-type theorems for stochastic functional differential equations[J]. Discrete And Continuous Dynamical Systems-Series B, 2013, 18(6): 1697-1714
[47] Gahinet G, Nemirovski A, Laub A J, et al. LMI control toolbox[M]. Mass, USA: The MathWorks Inc, 1995
[48] Huang L R, Mao X R. SMC design for robust H_∞ control of uncertain stochastic delay systems[J]. Automatica, 2010, 46(2): 405-412
[49] Huang L R. Memoryless SMC design methods for stochastic delay systems[J]. Asian J Control, 2012, 14(6): 1496-1504
[50] Niu Y G, Ho D W C, Lam L. Robust integral sliding mode control for uncertain stochastic systems with time-varying delay[J]. Automatica, 2005, 41(5): 873-880
[51] Niu Y G, Ho D W C, Wang X Y. Sliding mode control for Itô stochastic systems with Markovian switching[J]. Automatica, 2007, 43(10): 1784-1790
[52] Huang L R, Deng F Q. Razumikhin-type theorems on stability of neutral stochastic functional differential equations[J]. IEEE Trans Automatic Control, 2008, 53(7): 1718-1723
[53] Huang L R, Mao X R. On input-to-state stability of stochastic retarded systems with Markovian switching[J]. IEEE Trans Automatic Control, 2009, 54(8): 1898-1902
[54] Mao X R. Razumikhin-type theorems on exponential stability of stochastic functional differential equations[J]. Stoch Processes Appl, 1996, 65(2): 233-250
[55] Zhao P, Feng W, Kang Y. Stochastic input-to-state stability of switched stochastic nonlinear systems[J]. Automatica, 2012, 48(10): 2569-2576
[56] Huang L R, Mao X R. Stability of singular stochastic systems with Markovian switchings[J]. IEEE Trans Automatic Control, 2011, 56(2): 424-429
[57] Huang L R, Mao X R. On almost sure stability of hybrid stochastic systems with mode-dependent interval delays[J]. IEEE Trans Automatic Control, 2010, 55(8): 1946-1952
[58] Mao X R, Lam J, Huang L R. Stabilisation of hybrid stochastic differential equations by delay feedback control[J]. Systems & Control Letters, 2008, 57(11): 927-935
[59] Huang L R, Mao X R. Robust delayed-state-feeedback stabilization of uncertain stochastic systems[J]. Automatica, 2009, 45(5): 1332-1339
[60] Mao X R. Stabilization of continuous-time hybrid

stochastic differential equations by discrete-time feedback control[J].Automatica,2013,49(12):3677-3681

[61] Allwright J C,Astolfi A,Wong H P.A note on asymptotic stabilization of linear systems by periodic,piecewise constant,output feedback[J].Automatica,2005,41(2):339-344

[62] Chen T W,Francis B.Optimal sampled data control systems[M].London,UK:Springer-Verlag,1995

[63] Hagiwara T,Araki M.Design of stable state feedback controller based on the multirate sampling of the plant output [J].IEEE Trans Automatic Control,1988,33(9):812-819

[64] Bentsman J,Fakhfakh J,Lehman B.Vibrational stabilization of linear time delay systems and its robustness with respect to delay size[J].Systems & Control Letters,1989,12(3):267-272

[65] Lehman B,Bentsman J,Lunel S V,et al.Vibrational control of nonlinear time lag systems with bounded delay: Averaging theory,stabilizability,and transient behavior [J].IEEE Trans Automatic Control,1994,39(5):898-912

[66] Caraballo T,Robinson J C.Stabilization of linear PDEs by Strotonovich noise[J].Systems & Control Letters,2004,53(1):41-50

[67] Wu F K,Mao X R,Hu S G.Stochastic suppression and stabilization of functional differential equations [J].Systems & Control Letters,2010,59(12):745-753

[68] Bao J H,Yuan C G.Stabilization of Partial Differential Equations by Lévy noise[J].Stoch Anal Appl,2012,30(2):354-374

[69] Liu D Z,Wang W Q,Menaldi J L.Almost sure asymptotic stabilization of differential equations with time-varying delay by Lévy noise[J].Nonlinear Dynamics,2015,79(1):163-172

[70] Mao X R.A note on the LaSalle-type theorems for stochastic differential delay equations [J].J Math Anal Appl,2002,268(1):125-142

[71] Yuan C G,Mao X R.Robust stability and controllability of stochastic differential delay equations with Markovian switching[J].Automatica,2004,40(3):343-354

[72] Applebaum D,Siakalli M.Asymptotic stability of stochastic diffrential equations driven by Lévy noise[J].Journal of Applied Probability,2009,46(4):1116-1129

[73] Applebaum D,Siakalli M.Stochastic stabilization of dynamical systems using Lévy noise[J].Stochastics and Dynamics,2010,10(4):509-527

[74] Appleby J A D,Mao X R,Rodkina A.On stochastic stabilization of difference equations [J].Discrete and Continuous Dynamical Systems,2006,15(3):843-857

[75] Appleby J A D,Berkolaiko G,Rodkina A.Non-exponential stability and decay rates in nonlinear stochastic difference equations with unbounded noise[J].Stochastics,2009,81(2):99-127

[76] Rodkina A,Mao X R,Kolmanovskii V.On asymptotic behaviour of solutions of stochastic difference equations with Volterra type main term[J].Stochastic Anal Appl,2000,18(5):837-857

[77] Schmitendorf W E.Design of observer-based robust stabilizing controllers[J].Automatica,1988,24(5):693-696

[78] Sobel K M,Shapiro E Y.A design methodology for pitch pointing flight control systems[J].J Guid Control Dyn,1985,8(2):181-187

[79] Sathananthan S,Knap M J,Strong A,et al.Robust stability and stabilization of a class of nonlinear discrete time stochastic systems:An LMI approach[J].Appl Math Comput,2012,219(4):1988-1997

[80] Dragan V,Morozan T,Stoica A M.Mathematical methods in robust control of discrete-time linear stochastic systems [M].New York,USA:Springer,2009