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混杂随机微分方程理论概述

摘要

本文对混杂随机时滞微分方程的当前发展及最新的研究成果进行了概述.主要回顾了混杂随机微分方程解的存在唯一性定理、稳定性、鲁棒有界性以及离散时间反馈镇定性等部分对混杂随机微分方程的研究起到关键推动作用的成果.

关键词

布朗运动;马尔科夫链;存在唯一性;稳定性;反馈控制;镇定

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0 导读

本文原文为英文,希望感兴趣的读者进一步关注原文.

1)存在唯一性定理方面:对于一个给定的方程,它的解是否存在?如果解存在,其解是否唯一?这两个问题是研究一切稳定性问题及其他问题的基础,因此非常重要.经典结果要求方程系数满足局部利普希茨条件与线性增长条件,但很多高度非线性的混杂随机时滞方程并不满足线性增长条件.基于此,原文介绍了文献[17]的工作,该文献把经典的随机微分方程的 Khasminskii 判据拓展到混杂随机时滞方程,意味着在局部利普希茨条件和 Khasminskii 条件下,解的存在唯一性可以得到保证,而这两个条件允许方程系数具有高度非线性.

2)稳定性理论方面:处理时滞系统一般用 Lyapunov 泛函法,但是构造一个合适的 Lyapunov 泛函很困难,故原文概述的文献主要还是以 Lyapunov 函数法为主,而考虑的稳定性类型为指数稳定性.文献[14]给出了系统是指数稳定性的充分条件.文献[13]给出了 p-阶矩渐近稳定性的充分性判据;需要指出的是,由于系统是马尔科夫跳变的,因此文献[13]用了 M 矩阵方法,这个方法也是处理马尔科夫跳变系统常用的方法.文献[13]与[14]虽然给出了指数稳定性与矩渐近稳定性的充分条件,但是它们不能允许系统的系数是高阶非线性的.为此,文献[20]给出了 p-阶矩指数稳定的充分性条件,并且允许系数是高阶非线性的.后来,作者注意到,不同的模态可能导致 Lyapunov 函数的阶数不同,这样,文献[20]的结论就不再适用.因此,文献[17]又进行了新的探索,给出了混杂系统的均方指数稳定性的充分条件.

3)鲁棒稳定性与有界性方面:在关于系统渐近性态的研究中,鲁棒稳定性极受关注.然而,直到 2013 年,相关研究都是基于线性增长条件的,很少有文章关注不具备线性增长条件的非线性随机时滞系统的鲁棒稳定性.文献[31](即本文定理 10,11,13)首次建立了非线性混杂随机时滞微分方程的鲁棒稳定性的基本理论.文献[31](即本文定理 10,12)还给出了扰动系统的 p-阶矩渐近有界性.

4)离散时间反馈镇定方面:连续时间反馈需要知道状态在每个时刻的值,而离散时间反馈只需要知道固定时间点的状态值,因此采用离散时间反馈可以减小代价.在全局利普希茨条件下,只要离散反馈的周期足够小,那么离散时间反馈控制问题就可以转化为经典的连续时间控制问题.

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Hybrid stochastic differential delay equations

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Abstract This paper reviews the current developments in the study of hybrid stochastic differential delay equations (SDDEs) with emphasis on the existence-and-uniqueness theorems, boundedness, stability and stabilization.
Key words Brownian motion; Markov chain; existence and uniqueness; stability; feedback control; stabilization

0 Introduction

Time-delay is encountered in many real-world systems in science and industry. Differential delay equations(DDEs)(or more generally, functional differential equations) have been developed to model such time-delay systems. Time-delay often causes undesirable system transient response, or even instability. Stability of DDEs has hence been studied intensively for more than 50 years(see, e.g., [1-3]).

In 1980's, stochastic differential delay equations (SDDEs) were developed in order to model real-world systems which contain some uncertainties or are subject to external noises (see, e.g., [4-6]). Since then, stability has been one of the most important topics in the study of SDDEs. The literature in this area is huge and lots of papers are of open-access.

On the other hand, there has been increasing attention devoted to hybrid systems, in which continuous dynamics are intertwined with discrete events. One of the distinct features of such systems is that the underlying dynamics are subject to changes with respect to certain configurations. A convenient way of modelling is to use continuous-time Markov chains to delineate many practical systems where they may experience abrupt changes in their structure and parameters. Such hybrid systems have been considered for the modelling of electric power systems by [7] as well as for the control of a solar thermal central receiver by [8]. It was suggested by [9] to solve control-related issues in Battle Management Command, Control and Communications (BM/C^3) systems by using hybrid systems. Probabilistic structure and a two-time-scale approach for control of hybrid dynamic systems were examined by [10]. In addition, Markovian hybrid systems have also been used in emerging applications in financial engineering[11] and wireless communications[12].

An important class of hybrid systems is the class of hybrid SDDEs (also known as SDDEs with Markovian switching), which has been developed since 1990 to model real-world systems where they may experience abrupt changes in their structure and parameters in addition to time delays and uncertainties. One of the important issues in the study of hybrid SDDEs is the automatic control, with consequent emphasis being placed on the analysis of stability (see, e.g., [13-15]). The main aim of this paper is to review the current developments in the study of hybrid SDDEs and to show their new trends.

Throughout this paper, unless otherwise specified, we use the following notation. Let $|\cdot|$ be the Euclidean norm in $\mathbf{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^TA)}$ while its operator norm is denoted by $\|A\| = \sup_{|x|=1} Ax$. If A is a symmetric matrix, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote its largest and smallest eigenvalue. Moreover, let $\mathbf{R}_+ = [0,\infty)$ and $\tau>0$. Denote by $C([-\tau,0];\mathbf{R}^n)$ the family of continuous functions from $[-\tau,0]$ to $\mathbf{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\le\theta\le0} |\varphi(\theta)|$.

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). We will set $\mathcal{F}_\theta = \mathcal{F}_0$ for $\theta\in[-\tau,0]$. Let $B(t) = (B_1(t),\cdots,B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t\ge0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S=\{1,2,\cdots,N\}$ with generator $\Gamma=(\gamma_{ij})_{N\times N}$. Let $p>0$ and denote by $L^p_{\mathcal{F}_t}([-\tau,0];\mathbf{R}^n)$ the family of $\mathcal{F}_t$-measurable $C([-\tau,0];\mathbf{R}^n)$-valued random variables $\xi=\{\xi(\theta):-\tau\leqslant\theta\leqslant 0\}$ such that $\mathrm{E}\|\xi\|^p<\infty$. Denote by $L^p_{\mathcal{F}_t}(\Omega;\mathbf{R}^n)$ the family of $\mathcal{F}_t$-measurable $\mathbf{R}^n$-valued random variables X such that $\mathrm{E}|X|^p<\infty$.

In general, a hybrid SDDE has the form

$$\mathrm{d}x(t)=f(x(t),x(t-\tau_1(t)),\cdots,x(t-\tau_k(t)),r(t),t)\mathrm{d}t+ g(x(t),x(t-\tau_1(t)),\cdots,x(t-\tau_k(t)),r(t),t)\mathrm{d}B(t)$$

(see, e.g., [13-14]), but in this review we will only concentrate on its simpler form

$$\mathrm{d}x(t)=f(x(t),x(t-\tau),r(t),t)\mathrm{d}t+g(x(t),x(t-\tau),r(t),t)\mathrm{d}B(t)\quad \text{on } t\geqslant 0 \tag{1}$$

in order to avoid the notation becoming too complicated.

1 Existence-and-uniqueness theorems

Most of hybrid SDDEs used in many branches of science and industry are highly nonlinear. It is therefore important to know whether a given hybrid SDDE has a solution and whether the solution is unique.

Consider the nonlinear SDDE (1), where

$$f:\mathbf{R}^n\times\mathbf{R}^n\times S\times\mathbf{R}_+\to\mathbf{R}^n \quad\text{and}$$
$$g:\mathbf{R}^n\times\mathbf{R}^n\times S\times\mathbf{R}_+\to\mathbf{R}^{n\times m}$$

are Borel measurable. In order to solve the equation we need to know the initial data and we assume that they are given by

$$\{x(\theta):-\tau\leqslant\theta\leqslant 0\}=\xi\in C([-\tau,0];\mathbf{R}^n). \tag{2}$$

Of course, in general we may allow the initial data to be $C([-\tau,0];\mathbf{R}^n)$-valued $\mathcal{F}_0$-measurable random variables e.g. $\xi\in L^p_{\mathcal{F}_0}([-\tau,0];\mathbf{R}^n)$ but it is sufficient to consider (2) only in the contents of this review. Moreover, we also require the coefficients f and g to be sufficiently smooth. The well-known conditions imposed for the existence and uniqueness of the solution are the Local Lipschitz condition and the linear growth condition which are stated as follows.

Assumption 1 (The local Lipschitz condition) For each integer $h\geqslant 1$ there is a positive constant K_h such that

$$|f(x,y,i,t)-f(\bar{x},\bar{y},i,t)|^2\vee|g(x,y,i,t)-g(\bar{x},\bar{y},i,t)|^2\leqslant K_h(|x-\bar{x}|^2+|y-\bar{y}|^2)$$

for those $x,y,\bar{x},\bar{y}\in\mathbf{R}^n$ with $|x|\vee|y|\vee|\bar{x}|\vee|\bar{y}|\leqslant h$ and any $(i,t)\in S\times\mathbf{R}_+$.

Assumption 2 (The linear growth condition) There is a constant $K>0$ such that

$$|f(x,y,i,t)|^2\vee|g(x,y,i,t)|^2\leqslant K(1+|x|^2+|y|^2)$$

for all $(x,y,i,t)\in\mathbf{R}^n\times\mathbf{R}^n\times S\times\mathbf{R}_+$.

The following existence-and-uniqueness theorem is classical (see, e.g., [13-14]).

Theorem 1 Under Assumptions 1 and 2, equation (1) with the given initial data (2) has a unique continuous solution $x(t)$ on $t\geqslant-\tau$. Moreover, the solution has the property that for any $p>0$,

$$\mathrm{E}\Big|\Big(\sup_{-\tau\leqslant t\leqslant T}|x(t)|^p\Big)<\infty,\quad \forall T>0. \tag{3}$$

One important direction in the study of existence and uniqueness is to remove the linear growth condition because many hybrid SDDEs in practice do not obey it. For example, the following scalar delay Lotka-Volterra model

$$\mathrm{d}x(t)=x(t)[b(r(t))-a(r(t))x(t)]\mathrm{d}t+\sigma(r(t))x(t-\tau)\mathrm{d}B(t)$$

does not obey Assumption 2. It was in this spirit that Mao and Yuan[14] developed the classical Khasminskii test for SDEs[16] to cope with the hybrid SDDEs. To review this important result, we need more notation. Let $C(\mathbf{R}^n\times[-\tau,\infty);\mathbf{R}_+)$ denote the family of all continuous functions from $\mathbf{R}^n\times[-\tau,\infty)$ to $\mathbf{R}_+$. Denote by $C^{2,1}(\mathbf{R}^n\times S\times\mathbf{R}_+;\mathbf{R}_+)$ the family of all continuous nonnegative functions $V(x,i,t)$ defined on $\mathbf{R}^n\times S\times\mathbf{R}_+$ such that for each $i\in S$, they are continuously twice differentiable in x and once in t. Given $V\in C^{2,1}(\mathbf{R}^n\times S\times\mathbf{R}_+;\mathbf{R}_+)$, we define the function $LV:\mathbf{R}^n\times\mathbf{R}^n\times S\times\mathbf{R}_-\to\mathbf{R}$ by

$$LV(x,y,i,t)=V_t(x,i,t)+V_x(x,i,t)f(x,y,i,t)n+\frac{1}{2}\mathrm{trace}[g^{\mathrm{T}}(x,y,i,t)V_{xx}(x,i,t)g(x,y,i,t)]+\sum_{j=1}^{N}\gamma_{ij}V(x,j,t),$$

where

$$V_t(x,i,t)=\frac{\partial V(x,i,t)}{\partial t},$$
$$V_x(x,i,t)=\Big(\frac{\partial V(x,i,t)}{\partial x_1},\cdots,\frac{\partial V(x,i,t)}{\partial x_n}\Big),$$
$$V_{xx}(x,i,t)=\Big(\frac{\partial^2 V(x,i,t)}{\partial x_i\partial x_j}\Big)_{n\times n}.$$

Let us emphasize that LV is defined on $\mathbf{R}^n\times\mathbf{R}^n\times S\times\mathbf{R}_+$ while V on $\mathbf{R}^n\times S\times\mathbf{R}_+$. We can now state a Khasminkii-type theorem for hybrid SDDEs (see [14], Theorem 7.3 on page 280).

Theorem 2 Let Assumption 1 hold. Assume moreover that there are functions $V\in C^{2,1}(\mathbf{R}^n\times S\times\mathbf{R}_+;\mathbf{R}_+)$ and $U\in C(\mathbf{R}^n\times[-\tau,\infty);\mathbf{R}_+)$ as well as a positive constant α such that

$$\lim_{|x|\to\infty}\inf_{-\tau\leqslant t<\infty}U(x,t)=\infty,\tag{4}$$

$$U(x,t)\leqslant V(x,i,t),\tag{5}$$

$$LV(x,y,i,t)\leqslant\alpha[1+U(x,t)+U(y,t-\tau)]\tag{6}$$

for all $(x,y,i,t)\in\mathbf{R}^n\times\mathbf{R}^n\times S\times\mathbf{R}_+$. Then the SDDE (1) has a unique global solution on $t\in[-\tau,\infty)$ for any given initial data (2).

Let us now take a further step to review another recent result in this direction[17]. There are still many important hybrid SDDEs which are not covered by Theorem 2. For example, consider the hybrid stochastic delay power logistic model

$$\mathrm{d}x(t)=x(t)([a_{r(t)}+b_{r(t)}x(t-\tau)-c_{r(t)}x^2(t)]\mathrm{d}t+\sigma_{r(t)}x(t-\tau)\mathrm{d}B(t))\tag{7}$$

in population dynamics, where a_i, b_i etc. are all real numbers. This hybrid SDDE can be used to model the population growth of certain species. When we attempt to apply Theorem 2 we encount a new problem. To explain, if we choose $V(x,i,t)=\theta_i x^2$ with $\theta_i>0$, then LV has the form

$$LV(x,y,i,t)=2\theta_i x^2[a_i+b_i y-c_i x^2]+\theta_i\sigma_i^2x^2y^2+\sum_{j=1}^N\gamma_{ij}\theta_j x^2.\tag{8}$$

Here the polynomial $2\theta_i x^2[b_i y-c_i x^2]+\theta_i\sigma_i^2x^2y^2$ appears on the right-hand side and it has an order of 4 which is higher than the order of $V(x,i,t)=\theta_i x^2$, whence (6) could not be satisfied. The following generalised Khasminskii-type theorem established in [17] is particularly useful in this situation.

Theorem 3 Let Assumption 1 hold. Assume moreover that there are functions $V\in C^{2,1}(\mathbf{R}^n\times S\times\mathbf{R}_+;\mathbf{R}_+)$ and $U_1,U_2\in C(\mathbf{R}^n\times[-\tau,\infty);\mathbf{R}_+)$ as well as a positive constant α such that

$$\lim_{|x|\to\infty}\inf_{-\tau\leqslant t<\infty}U_1(x,t)=\infty,\tag{9}$$

$$U_1(x,t)\leqslant V(x,i,t)\leqslant U_2(x,t),\tag{10}$$

$$LV(x,y,i,t)\leqslant\alpha[1+U_1(x,t)+U_1(y,t-\tau)]-U_2(x,t)+U_2(y,t-\tau)\tag{11}$$

for all $(x,y,i,t)\in\mathbf{R}^n\times\mathbf{R}^n\times S\times\mathbf{R}_+$. Then the SDDE (1) has a unique global solution on $t\in[-\tau,\infty)$ for any given initial data (2).

This theorem was essentially proved in [17] but what we state above is slightly more general. In fact, we can even take a further step, as [18] did, to show:

Theorem 4 Theorem 3 still holds if (11) is replaced by

$$LV(x,y,i,t)\leqslant\alpha[1+U_1(x,t)+U_1(y,t-\tau)]-U_2(x,t)+\beta U_2(y,t-\tau)\tag{12}$$

with some constant $\beta>1$.

But we leave the details of the proof to the reader.

2 Stability

From now on we will let Assumption 1 be our standing hypothesis which we will not mention explicitly. There are various types of stochastic stability (see, e.g., [5,14,16]) for hybrid SDDEs but we only review some of them. For the purpose of stability, we assume that $f(0,0,i,t)=0$ and $g(0,0,i,t)=0$ so the hybrid SDDE (1) admits a trivial solution $x(t)=0$.

One of the most important stability concepts is the exponential stability. Here, the key idea is to show not only that some quantity decays with time, but also that the decay is exponential. Applying this idea to the p-th moment of the solution gives moment exponential stability. Similarly, asking for exponential decay for almost all sample paths leads to almost sure exponential stability. Such properties are highly-valued in many application areas. Hybrid SDDEs are a kind of stochastic functional differential equations. It is natural to employ the Lyapunov functionals rather than functions to study the stability. However, it appears to be more difficult to construct the Lyapunov functionals than the Lyapunov functions. It is therefore important to explore the possibility of using the method of Lyapunov functions to determine sufficient conditions for stability. One of the useful criteria on the exponential stability is the following result (see [4], Theorem 7.22 on page 290).

Theorem 5 Assume that there is a function $V\in C^{2,1}(\mathbf{R}^n\times S\times\mathbf{R}_+;\mathbf{R}_+)$ and positive constants $p,\lambda_1,\lambda_2,c_1,c_2$ with $\lambda_1>\lambda_2$ such that

$$c_1|x|^p\leqslant V(x,i,t)\leqslant c_2|x|^p,\quad\forall(x,i,t)\in\mathbf{R}^n\times S\times\mathbf{R}_+,\tag{13}$$

and

$$LV(x,y,i,t)\le -\lambda_1|x|^p+\lambda_2|y|^p,\quad \forall (x,y,i,t)\in \mathbf{R}^n\times\mathbf{R}^n\times S\times\mathbf{R}_+. \tag{14}$$

Then for every initial data (2), the solution of equation (1) obeys

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathrm{E}|x(t)|^p)\le -\lambda,$$

where $\lambda\in(0,\lambda_1-\lambda_2)$ is the unique root to $\lambda_2 e^{\lambda\tau}=\lambda_1-\lambda c_2$. That is, equation (1) is exponentially stable in p-th moment.

In the stability study, the method of M-matrices (see, e.g., [19]) has been used frequently. For illustration, let us cite a result from [13], which can be used very conveniently in practice.

Theorem 6 Let $p\ge 2$. Assume that for each $i\in S$, there is a pair of real numbers β_i and σ_i such that

$$x^{\mathrm{T}}f(x,y,i,t)+\frac{p-1}{2}|g(x,y,i,t)|^2\le\beta_i|x|^2+\sigma_i|y|^2 \tag{15}$$

for all $(x,y,t)\in\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}_+$. Assume that

$$\mathcal{A}:=-\mathrm{diag}(p\beta_1,\cdots,p\beta_N)-\Gamma$$

is a nonsingular M-matrix so $(q_1,\cdots,q_N)^{\mathrm{T}}:=\mathcal{A}^{-1}\overrightarrow{1}\gg 0$, where $\overrightarrow{1}=(1,1,\cdots,1)^{\mathrm{T}}$. If

$$pq_i\sigma_i<1,\quad \forall i\in S, \tag{16}$$

then the SDDE (1) is asymptotically stable in p-th moment.

To explain some special features which Theorem 6 shows, let us consider a simple scalar SDDE

$$\mathrm{d}x(t)=a(r(t))x(t)\mathrm{d}t+b(r(t))x(t-\tau)\mathrm{d}B(t), \tag{17}$$

where $B(t)$ is a scalar Brownian motion, $r(t)$ is a right-continuous Markov chain with the state space $S=\{1,2\}$ and the generator

$$\Gamma=(\gamma_{ij})_{2\times 2}=\begin{pmatrix}-1 & 1\\ \gamma & -\gamma\end{pmatrix},\quad \gamma>0$$

and

$$a(i)=\begin{cases}-6 & \text{if } i=1,\\ 1 & \text{if } i=2,\end{cases}\quad b(i)=\begin{cases}1 & \text{if } i=1,\\ 2 & \text{if } i=2.\end{cases}$$

The SDDE (17) can be regarded as the result of the following two SDDEs

$$\mathrm{d}x(t)=-6x(t)\mathrm{d}t+x(t-\tau)\mathrm{d}B(t) \tag{18}$$

and

$$\mathrm{d}x(t)=x(t)\mathrm{d}t+2x(t-\tau)\mathrm{d}B(t) \tag{19}$$

switching from one to the other according to the movement of the Markov chain $r(t)$. It is easy to see that the SDDE (18) is mean-square exponentially stable while the SDDE (19) is not. However, we shall see that if γ is sufficiently large, then the hybrid SDDE (17) will be mean-square exponentially stable. In fact, to apply Theorem 6 with $p=2$, we observe that (15) becomes

$$a(i)x^2+0.5b^2(i)y^2=\begin{cases}-6x^2+0.5y^2 & \text{if } i=1,\\ x^2+2y^2 & \text{if } i=2.\end{cases}$$

giving $\beta_1=-6$, $\beta_2=1$, $\sigma_1=0.5$ and $\sigma_2=2$. Accordingly, the matrix $\mathcal{A}$ defined in Theorem 6 becomes

$$\mathcal{A}=\begin{pmatrix}13 & -1\\ -\gamma & \gamma-2\end{pmatrix}.$$

This is an M-matrix provided $\gamma>13/6$. Compute

$$\mathcal{A}^{-1}=\frac{1}{12\gamma-25}\begin{pmatrix}\gamma-2 & 1\\ \gamma & 13\end{pmatrix},$$

whence

$$\begin{pmatrix}q_1\\ q_2\end{pmatrix}=\mathcal{A}^{-1}\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}(\gamma-1)/(12\gamma-26)\\ (\gamma+13)/(12\gamma-26)\end{pmatrix}.$$

Consequently, (16) becomes

$$\frac{\gamma-1}{12\gamma-26}<1\quad\text{and}\quad\frac{4(\gamma+13)}{12\gamma-26}<1,$$

which require $\gamma>3.25$. We can therefore conclude that the hybrid SDDE (17) will be mean-square exponentially stable provided $\gamma>3.25$.

We observe that Theorem 5 does not only require the function $V(x,i,t)$ have the same degree p for each $i\in S$ but also the diffusion operator $LV(x,y,i,t)$ be bounded by a polynomial with the same degree p for every $i\in S$. These requirements are somehow restrictive. For example, recalling equation (7) and its corresponding LV of the form (8), we see that Theorem 5 could not be applied as the higher order polynomial $2\theta_i x^2[b_i y-c_i x^2]+\theta_i\sigma_i^2x^2y^2$ appears in LV. To cope with this situation, the following theorem is useful.

Theorem 7 Assume that there are functions $V\in C^{2,1}(\mathbf{R}^n\times S\times\mathbf{R}_+;\mathbf{R}_+)$ and $U\in C(\mathbf{R}^n;\mathbf{R}_+)$, and positive constants $p,c_1,c_2,\lambda_1>\lambda_2$ and $\lambda_3>\lambda_4$ such that

$$c_1|x|^p\le V(x,i,t)\le c_2|x|^p,\quad \forall (x,i,t)\in\mathbf{R}^n\times S\times\mathbf{R}_+, \tag{20}$$

and

$$LV(x,y,i,t)\le-\lambda_1|x|^p+\lambda_2|y|^p-\lambda_3U(x)+\lambda_4U(y),\quad \forall (x,y,i,t)\in\mathbf{R}^n\times\mathbf{R}^n\times S\times\mathbf{R}_+. \tag{21}$$

Then equation (1) is exponentially stable in p th moment.

This theorem is a generalisation of Theorem 7 and can be proved by the technique developed in [22] though we leave the details to the reader. However, we are going to explain that this theorem is till not be able to cope with the hybrid SDDEs who have different nonlinear structures in different modes. For example, consider a scalar hybrid SDDE

$$\mathrm{d}x(t)=f(x(t),x(t-\tau),r(t),t)\mathrm{d}t+g(x(t),x(t-\tau),r(t),t)\mathrm{d}B(t). \quad (22)$$

Here $B(t)$ is a scalar Brownian motion, $r(t)$ is a right-continuous Markov chain on the state space $S=\{1,2\}$ with the generator

$$\Gamma=\begin{pmatrix}-1 & 1\\ 4 & -4\end{pmatrix}, \quad (23)$$

while

$$f(x,y,t,1)=-2(x+x^3), \quad g(x,y,t,1)=y^2,$$
$$f(x,y,t,2)=0.5x, \quad g(x,y,t,2)=0.5y,$$

for $(x,y,t)\in\mathbf{R}\times\mathbf{R}\times\mathbf{R}_+$. This hybrid SDDE can be regarded as it operates in two modes (namely mode 1 when $r(t)=1$ and mode 2 when $r(t)=2$) and it obeys

$$\mathrm{d}x(t)=-2[x(t)+x^3(t)]\mathrm{d}t+x^2(t-\tau)\mathrm{d}B(t) \quad (24)$$

and

$$\mathrm{d}x(t)=0.5x(t)\mathrm{d}t+0.5x(t-\tau)\mathrm{d}B(t) \quad (25)$$

in mode 1 and 2, respectively, and the system will switch from one mode to the other according to the probability law of the Markov chain. We observe that sub-system (24) in mode 1 is nonlinear while the other sub-system (25) in mode 2 is linear. That is, the hybrid SDDE (23) has significantly different structures in different modes. For such a hybrid SDDE, when we attempt to apply e.g. Theorem 7 to study its stability, we encounter a new problem. To see this new problem, we define, for example, $V\in C^{2,1}(\mathbf{R}\times S\times\mathbf{R}_+;\mathbf{R}_+)$ by

$$V(x,i,t)=\begin{cases}x^2, & \text{if } i=1,\\ \theta x^2, & \text{if } i=2,\end{cases}$$

for $(x,t)\in\mathbf{R}\times\mathbf{R}_+$, where θ is a positive number. Clearly,

$$(1\wedge\theta)x^2\leqslant V(x,i,t)\leqslant(1\vee\theta)x^2,$$

that is, condition (20) is fulfilled. However, compute

$$LV(x,y,1,t)=-4x(x+x^3)+y^4-x^2+\theta x^2=-(5-\theta)x^2-4x^4+y^4, \quad (26)$$

and

$$LV(x,y,2,t)=\theta x^2+0.25\theta y^2+4x^2-4\theta x^2=-(3\theta-4)x^2+0.25\theta y^2, \quad (27)$$

we observe that they are polynomials with different degrees in different modes whence condition (21) could not be satisfied. This problem prevents Theorem 7 from being used.

To tackle this new problem, we observe that the hybrid SDDE (22) has different nonlinear structures in different modes. It is natural to use different types of Lyapunov functions in different modes. For example, let us define $V\in C^{2,1}(\mathbf{R}\times S\times\mathbf{R}_+\mathbf{R}_+)$ by

$$V(x,i,t)=\begin{cases}x^2, & \text{if } i=1,\\ 2(x^2+x^4), & \text{if } i=2,\end{cases}$$

for $(x,t)\in\mathbf{R}\times\mathbf{R}_+$. This of course does not obey condition (13), but it will make LV to have a similar form as condition (14). In fact, compute

$$LV(x,y,1,t)=-4x(x+x^3)+y^4-x^2+2(x^2+x^4)=-3x^2-2x^4+y^4, \quad (28)$$

and

$$LV(x,y,2,t)=2x^2+4x^4+(0.5+3x^2)y^2+4x^2-8(x^2+x^4)\leqslant-2x^2+0.5y^2-1.75x^4+y^4. \quad (29)$$

That is, we always have

$$LV(x,y,i,t)\leqslant-1.75(x^2+x^4)+(y^2+y^4)$$

for $i=1,2$. If we define $U_1(x)=x^2$ and $U_2(x)=2(x^2+x^4)$ for $x\in\mathbf{R}$, then

$$U_1(x)\leqslant V(x,i,t)\leqslant U_2(x) \quad (30)$$

and

$$LV(x,y,i,t)\leqslant-0.875U_2(x)+0.5U_2(y) \quad (31)$$

for all $x,y\in\mathbf{R}$, $t\geqslant0$ and $i=1,2$. We observe that these are significantly weaker than conditions (20) and (21). The question is: can we establish the stability result under these weaker conditions? A positive answer to this question was established recently in [17]. The following is a theorem from [17] which shows that the hybrid SDDE (22) is exponentially stable in mean square.

Theorem 8 Assume that there are three functions $V\in C^{2,1}(\mathbf{R}^n\times S\times\mathbf{R}_+;\mathbf{R}_+)$ and $U_1,U_2\in C(\mathbf{R}^n\times[-\tau,\infty);\mathbf{R}_+)$, as well as two constants $\lambda_1>\lambda_2>0$, such that

$$\lim_{|x|\to\infty}(\inf_{t\geqslant0}U_1(x,t))=\infty, \quad (32)$$

$$U_1(x,t)\leqslant V(x,i,t)\leqslant U_2(x,t),\quad \forall(x,i,t)\in\mathbf{R}^n\times S\times\mathbf{R}_+, \quad (33)$$

$$LV(x,y,i,t)\leqslant-\lambda_1U_2(x,t)+\lambda_2U_2(y,t-\tau),\quad \forall(x,y,i,t)\in\mathbf{R}^n\times\mathbf{R}^n\times S\times\mathbf{R}_+. \quad (34)$$

Then the solution of equation (1) with the initial data

(2) has the moment properties that

$$\lim_{t\to\infty}\frac{1}{t}\sup\log(\mathrm{E}U_1(x(t),t))\leqslant-\varepsilon \tag{35}$$

$$\int_0^\infty \mathrm{E}U_2(x(t),t)\mathrm{d}t<\infty; \tag{36}$$

while it also has the sample (pathwise) properties that

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(U_1(x(t),t))\leqslant-\varepsilon \quad \text{a.s.} \tag{37}$$

$$\int_0^\infty U_2(x(t),t)\mathrm{d}t<\infty \quad \text{a.s.} \tag{38}$$

where $\varepsilon>0$ is the unique root to the equation $\lambda_1=\varepsilon+c\lambda_2 e^{\varepsilon\tau}$.

The asymptotic estimates obtained in Theorem 8 are in terms of functions U_1 and U_2. If we impose a bit more condition on these functions, we can obtain more explicit estimates on the solutions. For example, if we further assume that

$$c_1|x|^p\leqslant U_1(x,t) \quad \text{and}$$
$$c_2|x|^q\leqslant U_2(x,t) \quad \forall (x,t)\in\mathbf{R}^n\times\mathbf{R}_+ \tag{39}$$

for some positive numbers c_1, c_1, p and q, then the solution has the moment properties that

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(\mathrm{E}|x(t)|^p)\leqslant-\varepsilon, \tag{40}$$

$$\int_0^\infty \mathrm{E}|x(t)|^q\mathrm{d}t<\infty; \tag{41}$$

while it also has the sample (pathwise) properties that

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(|x(t)|)\leqslant-\frac{\varepsilon}{p} \quad \text{a.s.}, \tag{42}$$

$$\int_0^\infty |x(t)|^q\mathrm{d}t<\infty \quad \text{a.s.} \tag{43}$$

We observe that assertions (40) and (42) show the exponential stability in p th moment and probability one (almost surely) (see, e.g., [20]), respectively, while assertions(41) and (43) show the H_∞-stability (see, e.g., [21]).

Talking about the moment exponential stability and almost sure one, they do not imply each other in general. However, in many hybrid SDDEs, the former does imply the latter. The following theorem describes this situation[13].

Theorem 9 Assume that there is a $K>0$ such that

$$|f(x,y,i,t)|\vee|g(x,y,i,t)|\leqslant K(|x|+|y|),$$
$$\forall (x,y,i,t)\in\mathbf{R}^n\times\mathbf{R}^n\times S\times\mathbf{R}_+. \tag{44}$$

Let $p>0$ and $\lambda>0$. If

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(\mathbf{E}|x(t)|^p)\leqslant-\lambda,$$

then

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(|x(t)|)\leqslant-\frac{\lambda}{p} \quad \text{a.s.}$$

3 Robust boundedness and stability

In the study of asymptotic properties, the robust stability has received a great deal of attention. For example, the motivation of robust stability together with examples can be found in Ackermann[22], and an excellent discussion of the stability radii of linear systems with structured perturbations can be found in Hinrichsen and Pritchard[23-24]. Su[25] and Tseng et al.[26] discussed the robust stability for linear delay equations. In the aspect of robustness of stochastic stability, Haussmann[27] studied the robust stability for a linear system and Ichikawa[28] for a semilinear system. Mao et al.[29] discussed the robust stability of uncertain linear or semilinear SDDEs. Mao[30] investigated the stability of the stochastic delay interval system with Markovian switching. For the further development in this direction we refer the reader to the book[14] and the references therein. However, up to 2013, the underlying SDDEs are either linear or nonlinear with the linear growth condition, and little is known about the robust stability of nonlinear SDDEs without the linear growth condition.

Hu et al.[31] are the first to study the robust stability of nonlinear hybrid SDDEs without the linear growth condition. To explain their motivation, let us consider the scalar cubic ordinary differential equation (ODE)

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=-ax(t)-bx^3(t), \tag{45}$$

Where $a,b>0$. Noting that

$$\frac{\mathrm{d}(x^2(t))}{\mathrm{d}t}=-2ax^2(t)-2bx^4(t)\leqslant-2ax^2(t), \tag{46}$$

we see easily that

$$x^2(t)\leqslant x^2(0)e^{-2at}.$$

That is, equation (45) is exponentially stable. This equation has appeared frequently in engineering, biology, etc. and has been widely used to explain the concept of stability of nonlinear differential equations (see,e.g.,[32]).In practice,we need to estimate the system parameters,for instance, a. Given $x(t)$ at time T, we can estimate a as

$$\bar{a}+\text{random error},$$

where $\bar{a}$ is the average of a. We assume that a is positive and is independent of $x(t)$ in order to keep this example simple. By the well-known central limit theorem the random error can be represented by a normal distribution (more precisely, a white noise), that is, we may write a as

$$\bar{a}+\sqrt{V_{\mathrm{ar}}}\dot{B}(t),$$

where V_{ar} is the variance which may or may not depend on the state $x(t)$, and $\dot{B}(t)$ is a white noise (i.e., $B(t)$ is a scalar Brownian motion). If V_{ar} is dependent on $x(t)$, say $V_{\mathrm{ar}}=(\sigma x(t))^2$, then equation (45) becomes the Itô SDE

$$\mathrm{d}x(t)=[-\bar{a}x(t)-bx^3(t)]\mathrm{d}t+\sigma x^2(t)\mathrm{d}B(t). \quad (47)$$

This equation can of course be regarded as a stochastically perturbed system of equation (45) and the stochastic perturbation $\sigma x^2(t)\mathrm{d}B(t)$ is a nonlinear form of $x(t)$. By the Itô formula, we have

$$\mathrm{d}(x^2(t))=[-2\bar{a}x^2(t)-(2b-\sigma^2)x^4(t)]\mathrm{d}t+2\sigma x^3(t)\mathrm{d}B(t)\leqslant-2\bar{a}x^2(t)\mathrm{d}t+2\sigma x^3(t)\mathrm{d}B(t), \quad (48)$$

If $\sigma^2\leqslant 2b$. This implies easily that

$$\mathrm{E}(x^2(t))\leqslant x^2(0)\mathrm{e}^{-2\bar{a}t}.$$

In other words, the SDE (47) is exponentially stable in mean square. This shows that the given equation (45) can tolerate the nonlinear stochastic perturbation $\sigma x^2(t)\mathrm{d}B(t)$ so that its perturbed SDE (47) remains stable provided that the intensity of the stochastic perturbation is sufficiently small (i.e., $\sigma^2\leqslant 2b$).

Let us now take one more step. It is often that we have to estimate parameter a based on the past state, say $x(t-\tau)$, instead of the current state $x(t)$, due to the time lag. Hence the variance above may have the form $V_{\mathrm{ar}}=(\sigma x(t-\tau))^2$. Accordingly, equation (45) becomes the Itô SDDE

$$\mathrm{d}x(t)=[-\bar{a}x(t)-bx^3(t)]\mathrm{d}t+\sigma x(t)|x(t-\tau)|\mathrm{d}B(t). \quad (49)$$

Once again, this SDDE may be regarded as a stochastically perturbed system of equation (45) and the stochastic perturbation $\sigma x(t)|x(t-\tau)|\mathrm{d}B(t)$ is a nonlinear form of $x(t)$ and $x(t-\tau)$. By the Itô formula, we have

$$\mathrm{d}(x^2(t))\leqslant[-2\bar{a}x^2(t)-(2b-0.5\sigma^2)x^4(t)+0.5\sigma^2x^4(t-\tau)]\mathrm{d}t+2\sigma x^2(t)|x(t-\tau)|\mathrm{d}B(t). \quad (50)$$

Theorem 7 shows that this SDDE is exponentially stable in mean square provided $\sigma^2<2b$. In other words, the given equation (45) can also tolerate the nonlinear stochastic delay perturbation without losing stability property.

Let us now take one furthermore step in order to form a nonlinear perturbed hybrid SDDE. Assume the cubic system is operated in two modes. In mode 1, it obeys the ODE

$$\dot{x}(t)=-a(1)x(t)-b(1)x^3(t),$$

while, in mode 2, it obeys the ODE

$$\dot{x}(t)=-a(2)x(t)-b(2)x^3(t).$$

The system will switch from one mode to the other according to the Markov chain $r(t)$ on the state space $S=\{1,2\}$. In other words, the system is described by the hybrid ODE

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=-a(r(t))x(t)-b(r(t))x^3(t). \quad (51)$$

By the technique of parameter estimation, its stochastically perturbed system may take the following hybrid SDDE form

$$\mathrm{d}x(t)=[-a(r(t))x(t)-b(r(t))x^3(t)]\mathrm{d}t+\sigma(r(t))x(t)|x(t-\tau)|\mathrm{d}B(t), \quad (52)$$

where $\sigma(1)$ and $\sigma(2)$ are two numbers, representing the intensities of the stochastic perturbation. The study of robust stability is to get the bounds on both of them so that the perturbed hybrid SDDE (52) remains stable.

The examples above show clearly that a given system may subject to a nonlinear stochastic perturbation which do not satisfy the linear growth condition. In the simple case of (47) or (49), the existing theory shows that the corresponding perturbed system is exponentially stable in mean square. That is, a stable system can tolerate a nonlinear perturbation without losing stability property. Hu et al.[31] are the first to develop a general theory on the robust stability for nonlinear hybrid SDDEs. Let us now review their contributions. They first established new criteria on asymptotic boundedness and stability under the following assumption.

Assumption 3 Let $q>p\geqslant 2$ and assume that for each $i \in S$, there are nonnegative numbers $\beta_{i1}, \beta_{i3}, \beta_{i4}$, β_{i5} and a real number β_{i2} such that

$$x^{\mathrm{T}} f(x,y,i,t) + \frac{p-1}{2} |g(x,y,i,t)|^2 \leqslant \beta_{i1} + \beta_{i2}|x|^2 + \beta_{i3}|y|^2 - \beta_{i4}|x|^{q-p+2} + \beta_{i5}|y|^{q-p+2} \quad (53)$$

for all $(x,y,t) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}_+$, and

$$\mathcal{A} := \mathrm{diag}(p\beta_{12}, \cdots, p\beta_{N2}) - \Gamma$$

is a nonsingular M-matrix.

Condition (53) imposed in this assumption is purely motivated by the examples discussed above. There is no need to explain how the first three terms in the right hand side of (53) appear, as they are very standard. To see how the fourth and fifth terms may appear, we set $p=2$ and $q=4$ so (53) becomes

$$x^{\mathrm{T}} f(x,y,i,t) + \frac{1}{2} |g(x,y,i,t)|^2 n \leqslant \beta_{i1} + \beta_{i2}|x|^2 + \beta_{i3}|y|^2 - \beta_{i4}|x|^4 + \beta_{i5}|y|^4.$$

Comparing this with the drift terms in the right hand side of (50), we see how the terms of $\beta_{i4}|x|^4$ and $\beta_{i5}|y|^4$ may appear naturally.

By the theory of M-matrices, we observe that under Assumption 3, we have

$$(\theta_1, \cdots, \theta_N)^{\mathrm{T}} := \mathcal{A}^{-1} \overrightarrow{1} > 0, \quad (54)$$

that is, $\theta_i > 0$ for all $i \in S$, where $\overrightarrow{1} = (1, \cdots, 1)^{\mathrm{T}}$. With these notation and assumptions, Hu et al.[31] proved the following two theorems.

Theorem 10 Let Assumption 3 hold. Let $(\theta_1, \cdots, \theta_N)^{\mathrm{T}}$ be defined by (54). Assume also that

$$\max_{i \in S} p\theta_i \beta_{i3} < 1 \quad (55)$$

and

$$\min_{i \in S} \theta_i \beta_{i4} > \max_{i \in S} \theta_i \beta_{i5}. \quad (56)$$

Then for any given initial data (2), there is a unique global solution $x(t)$ to the hybrid SDDE (1) on $t \in [-\tau, \infty)$. Moreover, the hybrid SDDE (1) is asymptotically bounded in p th moment, that is, the solution satisfies

$$\limsup_{t \to \infty} \mathrm{E}|x(t)|^p \leqslant \frac{\alpha_1}{\varepsilon c_1}, \quad (57)$$

where $\varepsilon = \varepsilon_1 \wedge e\ \varepsilon_2$ while $\varepsilon_2 = \log(\alpha_4/\alpha_5)/\tau$ and $\varepsilon_1 > 0$ is the unique root to the following equation

$$\alpha_2 = \varepsilon_1 c_2 + \alpha_3 \mathrm{e}^{\varepsilon_1 \tau}, \quad (58)$$

and the parameters used above are defined as follows:

$$c_1 = \min_{i \in S} \theta_i, \quad c_2 = \max_{i \in S} \theta_i, \quad \delta_1 = \max_{i \in S} p\theta_i \beta_{i1},$$

$$\delta_3 = \max_{i \in S} p\theta_i \beta_{i3}, \quad \delta_4 = \min_{i \in S} p\theta_i \beta_{i4}, \quad \delta_5 = \max_{i \in S} p\theta_i \beta_{i5},$$

$$\delta = 0.5(1-\delta_3), \quad \alpha_1 = (2/p)\delta^{-(p-2)/2} \delta_1^{p/2},$$

$$\alpha_2 = 1 - \frac{(\delta+\delta_3)(p-2)}{p}, \quad \alpha_3 = \frac{2\delta_3}{p},$$

$$\alpha_4 = \delta_4 - \frac{\delta_5(p-2)}{q}, \quad \alpha_5 = \frac{\delta_5(q-p+2)}{q}.$$

Theorem 11 Let all the conditions of Theorem 10 hold with $\beta_{i1} = 0$ for all $i \in S$. Then for any given initial data (2), the unique global solution $x(t)$ to equation (1) has the properties that

$$\limsup_{t \to \infty} \frac{1}{t} \log(\mathrm{E}|x(t)|^p) \leqslant -\varepsilon \quad (59)$$

and

$$\limsup_{t \to \infty} \frac{1}{t} \log(|x(t)|) \leqslant -\frac{\varepsilon}{p} \quad \text{a.s.} \quad (60)$$

where $\varepsilon > 0$ is the same as defined in Theorem 10, and moreover,

$$\int_0^\infty \mathrm{E}|x(t)|^q \mathrm{d}t < \infty. \quad (61)$$

Hu et al.[31] then applied these two theorems to study the problems on the robust stability and boundedness proposed in the beginning of this section. To demonstrate their applications, let us consider an n-dimensional nonlinear hybrid differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(x(t), r(t), t), \quad (62)$$

Where $F: \mathbf{R}^n \times S \times \mathbf{R}_+ \to \mathbf{R}^n$. Assume that this given hybrid differential equation is either asymptotically stable or bounded. We would like to know how much a stochastic delay perturbation in the diffusion form that equation (62) can tolerate so that its perturbed system

$$\mathrm{d}x(t) = F(x(t), r(t), t)\mathrm{d}t + G(x(t-\tau), r(t), t)\mathrm{d}B(t) \quad (63)$$

remains asymptotically stable or bounded, where $G: \mathbf{R}^n \times S \times \mathbf{R}_+ \to \mathbf{R}^{n \times m}$. As a standing hypothesis, we assume that both F and G satisfy the local Lipschitz condition (namely Assumption 1).

Let us first discuss the robust boundedness. The following assumption guarantees the asymptotic boundedness of the given hybrid differential equation (62).

Assumption 4 Let $q > p \geqslant 2$ and assume that for each $i \in S$, there are nonnegative numbers $\bar{\beta}_{i1}, \bar{\beta}_{i4}$ and a real number $\bar{\beta}_{i2}$ such that

$$x^{\mathrm{T}}F(x,i,t)\leqslant\bar{\beta}_{i1}+\bar{\beta}_{i2}|x|^2-\bar{\beta}_{i4}|x|^{q-p+2} \quad (64)$$

for all $(x,t)\in\mathbf{R}^n\times\mathbf{R}_+$, and

$$\bar{\mathcal{A}}:=-\mathrm{diag}(p\bar{\beta}_{12},\cdots,p\bar{\beta}_{N2})-\Gamma$$

is a nonsingular M-matrix.

We also need another condition on the uncertain diffusion coefficient G.

Assumption 5 Let $q>p\geqslant2$ be the same as in Assumption 4 and assume that for each $i\in S$, there are nonnegative numbers $\bar{\beta}_{i3}$, $\bar{\beta}_{i5}$ and $\bar{\beta}_{i6}$ such that

$$|G(y,i,t)|^2\leqslant\bar{\beta}_{i6}+\bar{\beta}_{i3}|y|^2+\bar{\beta}_{i5}|y|^{q-p+2} \quad (65)$$

for all $(y,t)\in\mathbf{R}^n\times\mathbf{R}_+$.

The study of robust boundedness is to give the bounds on the parameters $\bar{\beta}_{i3}$ and $\bar{\beta}_{i5}$ in order for the perturbed system (63) to remain asymptotically bounded. The following theorem describes this situation.

Theorem 12 Let Assumptions 4 and 5 hold. Define

$$(\bar{\theta}_1,\cdots,\bar{\theta}_N)^{\mathrm{T}}:=\bar{\mathcal{A}}^{-1}\vec{1}, \quad (66)$$

where $\vec{1}$ has been defined before (so all $\bar{\theta}_i$'s are positive). If

$$\bar{\beta}_{i3}<\frac{2}{p(p-1)\bar{\theta}_i} \quad \text{and} \quad \bar{\beta}_{i5}<\frac{2\min_{j\in S}\bar{\theta}_j\bar{\beta}_{j4}}{(p-1)\bar{\theta}_i} \quad (67)$$

for all $i\in S$, then the perturbed system (63) is asymptotically bounded in p-th moment.

This theorem is a simple application of Theorem 10. Similarly but using Theorem 11, we can show the following theorem on the robust stability.

Theorem 13 Let Assumptions 4 and 5 hold with $\bar{\beta}_{i1}=\bar{\beta}_{i6}=0$ for all $i\in S$. Let $\bar{\theta}_i$'s be the same as defined in Theorem 12. If (67) holds, then the perturbed system (63) is not only exponentially stable in pth moment but also almost surely exponentially stable, and moreover, the solution of the perturbed system satisfies

$$\int_0^\infty \mathrm{E}|x(t)|^q\mathrm{d}t<\infty.$$

We next consider how much a delay perturbation in the drift part that equation (62) can tolerate so that its perturbed system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=F(x(t),r(t),t)+\hat{G}(x(t-\tau),r(t),t) \quad (68)$$

remains asymptotically stable or bounded, where $\hat{G}:\mathbf{R}^n\times S\times\mathbf{R}_+\to\mathbf{R}^n$. As a standing hypothesis, we assume as usual that both F and $\hat{G}$ satisfy the local Lipschitz condition (namely Assumption 1). Once again, let us discuss the robust boundedness first. Similarly (to Assumption 5), we impose the following assumption on the uncertain coefficient $\hat{G}$.

Assumption 6 Let $q>p\geqslant2$ and assume that for each $i\in S$, there are nonnegative numbers $\hat{\beta}_{i3}$, $\hat{\beta}_{i5}$ and $\hat{\beta}_{i6}$ such that

$$|\hat{G}(y,i,t)|^2\leqslant\hat{\beta}_{i6}+\hat{\beta}_{i3}|y|^2+\hat{\beta}_{i5}|y|^{q-p+2} \quad (69)$$

for all $(y,t)\in\mathbf{R}^n\times\mathbf{R}_+$.

Of course we need a similar condition on F as Assumption 4 which guarantees the asymptotic boundedness of the given hybrid differential equation (62). In order to obtain the explicit bounds on $\hat{\beta}_{i3}$, $\hat{\beta}_{i5}$ for the robust boundedness, we will split $\bar{\beta}_{i2}$ into $\hat{\beta}_{12}-\tilde{\beta}_{i2}$ with $\tilde{\beta}_{i2}>0$ as stated below.

Assumption 7 Let $q>p\geqslant2$ be the same as in Assumption 6 and assume that for each $i\in S$, there are nonnegative numbers $\bar{\beta}_{i1}$, $\bar{\beta}_{i4}$, a real number $\bar{\beta}_{i2}$ and a positive number $\tilde{\beta}_{i2}$ such that

$$x^{\mathrm{T}}F(x,i,t)\leqslant\hat{\beta}_{i1}+(\hat{\beta}_{i2}-\tilde{\beta}_{i2})|x|^2-\hat{\beta}_{i4}|x|^{q-p+2} \quad (70)$$

for all $(x,t)\in\mathbf{R}^n\times\mathbf{R}_+$, and

$$\hat{\mathcal{A}}:=-\mathrm{diag}(p\hat{\beta}_{12},\cdots,p\hat{\beta}_{N2})-\Gamma$$

is a nonsingular M-matrix.

The following result then follows from Theorem 10 easily.

Theorem 14 Let Assumptions 6 and 7 hold. Define

$$(\hat{\theta}_1,\cdots,\hat{\theta}_N)^{\mathrm{T}}:=\hat{\mathcal{A}}^{-1}\vec{1}, \quad (71)$$

where $\vec{1}$ has been defined before (so all $\hat{\theta}_i$'s are positive). If

$$\hat{\beta}_{i3}<\frac{4\tilde{\beta}_{i2}}{p\hat{\theta}_i} \quad \text{and} \quad \hat{\beta}_{i5}<\frac{4\tilde{\beta}_{i2}(\min_{j\in S}\hat{\theta}_j\hat{\beta}_{j4})}{\hat{\theta}_i} \quad (72)$$

for all $i\in S$, then the perturbed system (68) is asymptotically bounded in p th moment.

Similarly, we can show the following theorem on the robust stability.

Theorem 15 Let Assumptions 6 and 7 hold with $\hat{\beta}_{i1}=\hat{\beta}_{i6}=0$ for all $i\in S$. Let $\hat{\theta}_i$'s be the same as defined in Theorem 14. If (72) holds, then the perturbed system (68) is not only exponentially stable in p-th moment but also almost surely exponentially stable, and moreover, the solution satisfies $\int_0^\infty \mathrm{E}|x(t)|^q \mathrm{d}t<\infty$.

We of course could consider the perturbations in both drift and diffusion and the perturbations may not only depend on the past state $x(t-\tau)$ but also the current state $x(t)$. That is, the stochastically perturbed system of equation (62) may have the form

$$\mathrm{d}x(t)=F(x(t),r(t),t)\mathrm{d}t+G(x(t),x(t-\tau),r(t),t)\mathrm{d}B(t), \quad (73)$$

or an even more general form

$$\mathrm{d}x(t)=[F(x(t),r(t),t)+\hat{G}(x(t),x(t-\tau),r(t),t)]\mathrm{d}t+G(x(t),x(t-\tau),r(t),t)\mathrm{d}B(t). \quad (74)$$

However, we leave the details to the reader.

4 Stabilization by discrete-time feedback controls

Given an unstable hybrid SDE in the form of (75) with $u=0$, it is required to find a feedback control $u(x(t),r(t),t)$, based on the current state, so that the controlled system

$$\mathrm{d}x(t)=(f(x(t),r(t),t)+u(x(t),r(t),t))\mathrm{d}t+g(x(t),r(t),t)\mathrm{d}B(t) \quad (75)$$

becomes stable. Here the control u is $\mathbf{R}^n$-valued and, in practice, some of its components are set to be zero if the corresponding components of $\mathrm{d}x(t)$ are not affected by u. Such a continuous-time feedback control requires continuous observation of the state $x(t)$ for all time $t\geqslant 0$. However, it is more realistic and costs less in practice if the state is only observed at discrete times, say $0,\tau,2\tau,\cdots$, where $\tau>0$ is the duration between two consecutive observations. Accordingly, the feedback control should be designed based on these discrete-time observations, namely the feedback control should be of the form $u(x([t/\tau]\tau),r(t),t)$, where $[t/\tau]$ is the integer part of t/τ. Hence, the stabilization problem becomes to design a feedback control $u(x([t/\tau]\tau),r(t),t)$, based on discrete-time observations of the state, in the drift part so that the controlled system

$$\mathrm{d}x(t)=(f(x(t),r(t),t)+u(x([t/\tau]\tau),r(t),t))\mathrm{d}t+g(x(t),r(t),t)\mathrm{d}B(t) \quad (76)$$

becomes stable. Mao[33] is the first to study this stabilization problem. The theory in [33] shows that if the SDE(75) is mean-square exponentially stable, then so is the discrete-time controlled system(76) provided τ is sufficiently small. The theory enables us to transfer the discrete-time controlled problem into the classical (or regular) continuous-time controlled problem. Let us begin to review this theory. In [33], the following global Lipschitz condition is imposed.

Assumption 8 Assume that there are positive constants K_1,K_2,K_3 such that

$$\begin{aligned}|f(x,i,t)-f(y,i,t)|&\leqslant K_1|x-y|,\\ |u(x,i,t)-u(y,i,t)|&\leqslant K_2|x-y|,\\ |g(x,i,t)-g(y,i,t)|&\leqslant K_3|x-y|\end{aligned} \quad (77)$$

for all $(x,y,i,t)\in\mathbf{R}^n\times\mathbf{R}^n\times S\times\mathbf{R}_+$. Moreover,

$$f(0,i,t)=0,\quad u(0,i,t)=0,\quad g(0,i,t)=0 \quad (78)$$

for all $(i,t)\in S\times\mathbf{R}_+$.

We also observe that equation (76) is in fact a stochastic differential delay equation (SDDE) with a bounded variable delay. Indeed, if we define the bounded variable delay $\zeta:[t_0,\infty)\to[0,\tau]$ by

$$\zeta(t)=t-k\tau \quad \text{for} \quad k\tau\leqslant t<(k+1)\tau,$$

for $k=0,1,2,\cdots$, then equation (76) can be written as

$$\mathrm{d}x(t)=(f(x(t),r(t),t)+u(x(t-\zeta(t)),r(t),t))\mathrm{d}t+g(x(t),r(t),t)\mathrm{d}B(t). \quad (79)$$

It is therefore known (see, e.g., [14]) that under Assumption 8, equation (76) has a unique solution $x(t)$ such that $\mathrm{E}|x(t)|^2<\infty$ for all $t\geqslant 0$. Let us now introduce the auxiliary controlled hybrid SDE

$$\mathrm{d}y(t)=(f(y(t),r(t),t)+u(y(t),r(t),t))\mathrm{d}t+g(y(t),r(t),t)\mathrm{d}B(t) \quad (80)$$

on $t\geqslant t_0$, with initial data $y(t_0)=x_0\in\mathbf{R}^n$ and $r(t_0)=r_0\in S$ at time $t_0\geqslant 0$. The difference between this SDE and the original SDE (76) is that the feedback control here is based on the continuous-time state observation $y(t)$. Denoted by $y(t;x_0,r_0,t_0)$ the unique solution. Assume that we know how to design the control function $u:\mathbf{R}^n\times S\times\mathbf{R}_+\to\mathbf{R}^n$ for this auxiliary controlled hybrid SDE to be mean-square exponentially stable. (The techniques developed in [34-36], for example, can be used to design the control function u.) We will then show that this same control function also makes the original discrete-time controlled system (76) to be mean-square exponentially stable as long as τ is sufficiently small (namely we make state observations frequently enough). We therefore assume that this auxiliary controlled hybrid SDE (80) is mean-square exponentially stable. To be precise, let us state it as an assumption.

Assumption 9 Assume that there is a pair of positive constants M and γ such that the solution of the auxiliary controlled hybrid SDE (80) satisfies

$$\mathrm{E}|y(t;x_0,r_0,t_0)|^2 \leqslant M|x_0|\mathrm{e}^{-\gamma(t-t_0)} \quad \forall t \geqslant t_0 \quad (81)$$

for all $t_0 \geqslant 0$, $x_0 \in \mathbf{R}^n$ and $r_0 \in S$.

The key result of [33] is the following theorem.

Theorem 16 Let Assumptions 8 and 9 hold. Let $\tau^* > 0$ be the unique root to the equation

$$\bar{K}(\tau^*)(4M)^{(2K_1+3K_2+K_3^2)/\gamma} = \frac{1}{2}, \quad (82)$$

where

$$\bar{K}(\tau) = \frac{K_2 M\tau}{\gamma}[4\tau(K_1^2+K_2^2)+2K_3^2]\mathrm{e}^{(\gamma+2K_1+3K_2+K_3^2)\tau}. \quad (83)$$

If $\tau < \tau^*$, then there is a pair of positive constants $\bar{M}$ and λ such that the solution of the controlled hybrid SDE (76) satisfies

$$\mathrm{E}|x(t;x_0,r_0)|^2 \leqslant \bar{M}|x_0|^2\mathrm{e}^{-\lambda t}, \quad \forall t \geqslant 0 \quad (84)$$

for all $x_0 \in \mathbf{R}^n$ and $r_0 \in S$.

This is of course a very general result. However, it is due to the general technique used in 33 that the bound τ^* on τ is not very sharp. From the point of control cost, it is clearly better to have a larger τ^*. Influenced by [33], a number of recent papers (e.g., [37]) have significantly improved the bound τ^*. In particular, You et al.[38] have used the method of the Lyapunov functionals to make a very nice progress in this direction. Let us review their new techniques and results by stating their assumptions first.

Assumption 10 Assume that the coefficients f and g are all locally Lipschitz continuous (see Assumption 10). Moreover, they satisfy the following linear growth condition

$$|f(x,i,t)| \leqslant K_1|x| \quad \text{and} \quad |g(x,i,t)| \leqslant K_2|x| \quad (85)$$

for all $(x,i,t) \in \mathbf{R}^n \times S \times \mathbf{R}_+$, where both K_1 and K_2 are positive numbers.

Assumption 11 Assume that there exists a positive constant K_3 such that

$$|u(x,i,t)-u(y,i,t)| \leqslant K_3|x-y| \quad (86)$$

for all $(x,y,i,t) \in \mathbf{R}^n \times \mathbf{R}^n \times S \times \mathbf{R}_+$. Moreover, $u(0,i,t)=0$ for all $(i,t) \in S \times \mathbf{R}_+$.

The key technique in [38] is to use a Lyapunov functional on the segments $\hat{x}_t := \{x(t+s): -2\tau \leqslant s \leqslant 0\}$ and $\hat{r}_t := \{r(t+s): -2\tau \leqslant s \leqslant 0\}$ for $t \geqslant 0$. For $\hat{x}_t$ and $\hat{r}_t$ to be well defined for $0 \leqslant t < 2\tau$, they set $x(s) = x_0$ and $r(s) = r_0$ for $-2\tau \leqslant s \leqslant 0$. The Lyapunov functional used in [38] is of the form

$$V(\hat{x}_t,\hat{r}_t,t) = U(x(t),r(t),t) + \theta\int_{t-\tau}^{t}\int_{s}^{t}[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2 + |g(x(v),r(v),v)|^2]\mathrm{d}v\mathrm{d}s \quad (87)$$

for $t \geqslant 0$, where θ is a positive number to be determined later and we set

$$f(x,i,s)=f(x,i,0), \quad u(x,i,s)=u(x,i,0), \quad g(x,i,s)=f(x,i,0)$$

for $(x,i,s) \in \mathbf{R}^n \times S \times [-2\tau,0)$. Of course, the functional above uses $r(u)$ only on $t-\tau \leqslant u \leqslant t$ so we could have defined $\hat{r}_t := \{r(t+s): -\tau \leqslant s \leqslant 0\}$. But, to be consistent with the definition of $\hat{x}_t$, we define $\hat{r}_t$ as above and this does not lose any generality. We also require $U \in C^{2,1}(\mathbf{R}^n \times S \times \mathbf{R}_+; \mathbf{R}_+)$, the family of non-negative functions $U(x,i,t)$ defined on $(x,i,t) \in \mathbf{R}^n \times S \times \mathbf{R}_+$ which are continuously twice differentiable in x and once in t. For $U \in C^{2,1}(\mathbf{R}^n \times S \times \mathbf{R}_+; \mathbf{R}_+)$, let us define $\mathbb{L}U: \mathbf{R}^n \times S \times \mathbf{R}_+ \to \mathbf{R}$ by

$$\mathbb{L}U(x,i,t) = U_t(x,i,t) + U_x(x,i,t)[f(x,i,t)+u(x,i,t)] + \frac{1}{2}\mathrm{trace}[g^{\mathrm{T}}(x,i,t)U_{xx}(x,i,t)g(x,i,t)] + \sum_{j=1}^{N}\gamma_{ij}U(x,j,t). \quad (88)$$

The critical condition used in [38] is the following assumption on U.

Assumption 12 Assume that there is a function $U \in C^{2,1}(\mathbf{R}^n \times S \times \mathbf{R}_+; \mathbf{R}_+)$ and two positive numbers λ_1, λ_2 such that

$$\mathbb{L}U(x,i,t)+\lambda_1|U_x(x,i,t)|^2 \leqslant -\lambda_2|x|^2 \quad (89)$$

for all $(x,i,t) \in \mathbf{R}^n \times S \times \mathbf{R}_+$.

Let us comment on this assumption. Condition (89) implies

$$\mathbb{L}U(x,i,t) \leqslant -\lambda_2|x|^2, \quad (90)$$

which guarantees the asymptotic stability (in mean square etc.) of the controlled system (75). In other words, the continuous-time feedback control $u(x(t), r(t), t)$ will stabilize the system. However, in order for the discrete-time feedback control $u(x([t/\tau]\tau), r(t), t)$ to do the job, we need a slightly stronger condition, namely we add a new term $\lambda_1 |U_x(x,i,t)|^2$ into the left-hand-side of (90) to form (89). As demonstrated in [38], this is quite easy to achieve by choosing λ_1 sufficiently small when the derivative vector $U_x(x,i,t)$ is bounded by a linear function of x. We can now state the first result in [38].

Theorem 17 Let Assumptions 10, 11 and 12 hold. If $\tau>0$ is sufficiently small for

$$\lambda_2 > \frac{\tau K_3^2}{\lambda_1}[2\tau(K_1^2+2K_3^2)+K_2^2] \quad \text{and} \quad \tau \leqslant \frac{1}{4K_3}, \qquad (91)$$

then the controlled system (76) is H_∞-stable in the sense that

$$\int_0^\infty \mathrm{E}|x(s)|^2 \mathrm{d}s < \infty. \qquad (92)$$

for all initial data $x_0 \in \mathbf{R}^n$ and $r_0 \in S$.

In general, it does not follow from (92) that $\lim_{t\to\infty} \mathrm{E}(|x(t)|^2) = 0$. But, under the conditions imposed in [38], You et al. made this possible. This is their second result stated below.

Theorem 18 Under the same assumptions of Theorem 17, the solution of the controlled system (76) satisfies

$$\lim_{t\to\infty} \mathrm{E}|x(t)|^2 = 0$$

for all initial data $x_0 \in \mathbf{R}^n$ and $r_0 \in S$. That is, the controlled system (76) is asymptotically stable in mean square.

In general, one cannot imply $\lim_{t\to\infty}|x(t)| = 0$ a.s. from $\lim_{t\to\infty}\mathrm{E}(|x(t)|^2) = 0$. But, in their case, You et al.[38] once again made it possible. This is their third result.

Theorem 19 Under the same assumptions of Theorem 17, the solution of the controlled system (76) satisfies

$$\lim_{t\to\infty} x(t) = 0 \quad \text{a.s.}$$

for all initial data $x_0 \in \mathbf{R}^n$ and $r_0 \in S$. That is, the controlled system (76) is almost surely asymptotically stable.

The previous three theorems are on various asymptotic stabilities by feedback controls based on discrete-time state observations. However, all these stabilities do not reveal the rate at which the solution tends to zero. You et al.[38] then took a step further to discuss the exponential stabilization by feedback controls. For this purpose, they imposed another condition.

Assumption 10 Assume that there is a pair of positive numbers c_1 and c_2 such that

$$c_1|x|^2 \leqslant U(x,i,t) \leqslant c_2|x|^2 \qquad (93)$$

for all $(x,i,t) \in \mathbf{R}^n \times S \times \mathbf{R}_+$.

The following theorem from [38] shows that the controlled system (76) can be stabilized in the sense of both mean square and almost sure exponential stability.

Theorem 20 Let Assumptions 10, 11, 12 and 13 hold. Let $\tau > 0$ be sufficiently small for (91) to hold and set

$$\theta = \frac{K_3^2}{\lambda_1} \quad \text{and} \quad \lambda = \lambda_2 - \theta\tau[2\tau(K_1^2+2K_3^2)+K_2^2]$$

(so $\lambda > 0$). Then the solution of the controlled system (76) satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathrm{E}|x(t)|^2) \leqslant -\gamma \qquad (94)$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \leqslant -\frac{\gamma}{2} \quad \text{a.s.} \qquad (95)$$

for all initial data $x_0 \in \mathbf{R}^n$ and $r_0 \in S$, where $\gamma>0$ is the unique root to the following equation

$$2\tau\gamma \mathrm{e}^{2\tau\gamma}(H_1+\tau H_2)+\gamma c_2 = \lambda, \qquad (96)$$

in which

$$H_1 = \theta\tau(2\tau(K_1^2+2K_3^2)+K_2^2) + \frac{24\tau^3K_3^4}{1-6\tau^2K_3^2},$$

$$H_2 = \frac{12\theta\tau^2K_3^2(\tau K_1^2+K_2^2)}{1-6\tau^2K_3^2}. \qquad (97)$$

The use of Theorem 17 etc. depends on Assumptions 10, 11, 12 and 13. Among these, Assumption 12 is the critical one as the others can be verified easily. In other words, it is critical if we can design a control function $u(x,i,t)$ which satisfies Assumption 11 so that we can then further find a Lyapunov function $U(x,i,t)$ that fulfills Assumption 12.

It is known that the stabilization problem (75) by the continuous-time (regular) feedback control has been discussed by several authors e.g.[34-35].That is, to a certain degree, we know how to design a control function $u(x,i,t)$ which satisfies Assumption 11 so that we can then further find a Lyapunov function $U(x,i,t)$ that obeys (90).If the derivative vector $U_x(x,i,t)$ of this Lyapunov function is bounded by a linear function of x, we can then verify Assumption 12.This motivates us to propose the following alternative assumption.

Assumption 14 Assume that there is a function $U\in C^{2,1}(\mathbf{R}^n\times S\times\mathbf{R}_+;\mathbf{R}_+)$ and two positive numbers λ_3, λ_4 such that

$$\mathbb{L}U(x,i,t)\leqslant-\lambda_3|x|^2 \tag{98}$$

and

$$|U_x(x,i,t)|\leqslant\lambda_4|x| \tag{99}$$

for all $(x,i,t)\in\mathbf{R}^n\times S\times\mathbf{R}_+$.

In this case, if we choose a positive number $\lambda_1<\lambda_3/\lambda_4^2$, then

$$\mathbb{L}U(x,i,t)+\lambda_1|U_x(x,i,t)|^2\leqslant-(\lambda_3-\lambda_1\lambda_4^2)|x|^2. \tag{100}$$

But this is the desired condition (89) if we set $\lambda_2=\lambda_3-\lambda_1\lambda_4^2$. In other words, we have shown that Assumption 14 implies Assumption 12.

In practice, we often use the quadratic functions as the Lyapunov functions. That is, we use $U(x,i,t)=x^{\mathrm{T}}Q_ix$, where Q_i's are all symmetric positive-definite $n\times n$ matrices. In this case, Assumption 13 holds automatically with $c_1=\min_{i\in S}\lambda_{\min}(Q_i)$ and $c_2=\max_{i\in S}\lambda_{\max}(Q_i)$. Moreover, condition (99) holds as well with $\lambda_4=2\max_{i\in S}\|Q_i\|$. So all we need is to find Q_i's for (98) to hold. This motivates us to propose the following another assumption.

Assumption 15 Assume that there are symmetric positive-definite matrices $Q_i\in\mathbf{R}^{n\times n}$ $(i\in S)$ and a positive number λ_3 such that

$$2x^{\mathrm{T}}Q_i[f(x,i,t)+u(x,i,t)]+\mathrm{trace}[g^{\mathrm{T}}(x,i,t)Q_i(x,i,t)g(x,i,t)]+\sum_{j=1}^{N}\gamma_{ij}x^{\mathrm{T}}Q_jx\leqslant-\lambda_3|x|^2, \tag{101}$$

for all $(x,i,t)\in\mathbf{R}^n\times S\times\mathbf{R}_+$.

The following corollary follows immediately from Theorem 20.

Corollary 1 Let Assumptions 10, 11 and 15 hold. Set

$$c_1=\min_{i\in S}\lambda_{\min}(Q_i),\quad c_2=\max_{i\in S}\lambda_{\max}(Q_i),$$

$$\lambda_4=2\max_{i\in S}\|Q_i\|.$$

Choose $\lambda_1<\lambda_3/\lambda_4^2$ and then set $\lambda_2=\lambda_3-\lambda_1\lambda_4^2$. Let $\tau>0$ be sufficiently small for (91) to hold and set

$$\theta=\frac{K_3^2}{\lambda_1}\quad\text{and}\quad\lambda=\lambda_2-\theta\tau[2\tau(K_1^2+2K_3^2)+K_2^2]$$

(so $\lambda>0$). Then the assertions of Theorem 20 hold.

To close this section, let us take a further step. Observe that the discrete-time feedback control in the controlled SDE (76) is based on the discrete-time observations of the state, $x(k\tau)$ $(k=0,1,2,\cdots)$ but it still depends on the continuous-time observations of the mode, $r(t)$, on $t\geqslant0$. Of course this is perfectly fine if the mode of the system is obvious (i.e., fully observable at no cost), for example, in a financial system where the mode represents the interest rate (see, e.g., [11,39]). However, it could often be the case where the mode is not obvious and it costs to identify the current mode of the system (see, e.g., [12,40]). To reduce the control cost, it is reasonable that we identify the mode at the same discrete times $k\tau$ $(k\geqslant0)$ when we make observations for the state. It is in this spirit, Song et al.[41] recently consider an n-dimensional controlled hybrid system

$$\mathrm{d}x(t)=[f(x(t),r(t),t)+u(x([t/\tau]\tau),r([t/\tau]\tau),t)]\mathrm{d}t+g(x(t),r(t),t)\mathrm{d}B(t) \tag{102}$$

on $t\geqslant0$, where the new feedback control is based on the discrete-time observations of both state $x(k\tau)$ and mode $r(k\tau)$. They compare the controlled system (102) with the continuous-time controlled system (75) and show that if system (75) is mean-square exponentially stable, then so is system (102) provided τ is sufficiently small. The following theorem is their main result.

Theorem 21 Let Assumptions 8 and 9 hold. Let $\varepsilon\in(0,1)$ be a free parameter. Let $\bar{\tau}>0$ be the unique root to the equation

$$\overline{H}(\bar{\tau},\varepsilon)=0.5(1-\varepsilon), \tag{103}$$

where

$$\overline{H}(\tau,\varepsilon)=\frac{\tau K_2[4\tau(K_1^2+K_2^2)+2K_3^2]+4K_2(1-\mathrm{e}^{-\hat{\gamma}\tau})}{2K_1+2K_2+K_3^2}\times \mathrm{e}^{(2K_1+4K_2+K_3^2)(\tau+\log(2M/\varepsilon)/\gamma)}\times[\mathrm{e}^{(2K_1+2K_2+K_3^2)(\tau+\log(2M/\varepsilon)/\gamma)}-1], \tag{104}$$

in which

$$\hat{\gamma}=\max_{i\in S}(-\gamma_{ii}). \quad (105)$$

If $\tau<\bar{\tau}$, then there is a pair of positive constants $\bar{M}$ and λ such that the solution of the controlled hybrid system (102) satisfies

$$E|x(t;x_0,r_0)|^2\leqslant\bar{M}\,E|x_0|^2e^{-\lambda t} \quad \forall t\geqslant 0 \quad (106)$$

for all $x_0\in\mathbf{R}^n$ and $r_0\in S$. That is, the controlled hybrid system(102) is mean-square exponentially stable.

It should be pointed out that the results on the controlled system (102) are so far much fewer than these on the controlled system (76). The reader may wish to tackle it.

5 Conclusion

In this paper we made a review on the current developments in the study of hybrid SDDEs. We mainly focused on the existence-and-uniqueness theorems, asymptotic boundedness, stability and stabilization. Although these topics are among the most popular ones, there are some other important topics which we did not discuss, for example, the numerical methods for the hybrid SDDEs, the applications of hybrid SDDEs in various branches of science and industry.

References

[1] Fridman E. Introduction to time-delay systems: Analysis and control[M]. Birkhauser, 2014
[2] Hale J, K, Lunel S M. Introduction to functional differential equations[M]. Springer-Verlag, 1993
[3] Kolmanovskii V B, Nosov V R. Stability of functional differential equations[M]. Academic Press, 1986
[4] Ladde G S, Lakshmikantham V. Random differential inequalities[M]. Academic Press, 1980
[5] Mao X. Stochastic differential equations and their applications[M]. 2nd Edition, Elsevier, 2007
[6] Mohammed S-E A. Stochastic functional differential equations[M]. Longman Scientific and Technical, 1984
[7] Willsky A S, Levy B C. Stochastic stability research for complex power systems[R]. DOE Contract, LIDS, MIT, Rep. ET-76-C-01-2295, 1979
[8] Sworder D D, Rogers R O. An LQ-solution to a control problem associated with solar thermal central receiver [J]. IEEE Trans Automat Contr, 1983, 28: 971-978
[9] Athans M. Command and control (C2) theory: A challenge to control science[J]. IEEE Trans Automat Contr, 1987, 32: 286-293
[10] Yin G, Zhang Q. Continuous-time markov chains and applications: A singular perturbation approach [M]. Springer-Verlag, New York, 1998
[11] Yin G, Liu R H, Zhang Q. Recursive algorithms for stock Liquidation: A stochastic optimization approach[J]. SIAM J Optim, 2002, 13: 240-263
[12] Krishnamurthy V, Wang X, Yin G. Spreading code optimization and adaptation in CDMA via discrete stochastic approximation[J]. IEEE Trans Inform Theory, 2004, 50: 1927-1949
[13] Mao X, Matasov A, Piunovskiy A. Stochastic differential delay equations with Markovian switching[J]. Bernoulli, 2000, 6(1): 73-90
[14] Mao X, Yuan C. Stochastic Differential equations with markovian switching[M]. Imperial College Press, 2006
[15] Shaikhet L. Stability of stochastic hereditary systems with Markov switching[J]. Theory of Stochastic Processes, 1996, 2(18): 180-184
[16] Khasminskii R Z. Stochastic stability of differential equations[M]. Sijthoff and Noordhoff, 1981
[17] Hu L, Mao X, Shen Y. Stability and boundedness of nonlinear hybrid stochastic differential delay equations[J]. Systems and Control Letters, 2013, 62: 178-187
[18] Mao X, Rassias M J. Khasminskii-type theorems for stochastic differential delay equations[J]. J Sto Anal Appl, 2005, 23: 1045-1069
[19] Berman, A, Plemmons R J. Nonnegative matrices in the mathematical sciences[M]. SIAM, 1994
[20] Luo Q, Mao X, Shen Y. New criteria on exponential stability of neutral stochastic differential delay equations [J]. Systems and Control Letters, 2006, 55: 826-834
[21] Xu S, Lam J, Mao X, et al. A new LMI condition for delay-dependent robust stability of stochastic time-delay systems[J]. Asian Jornal of Control, 2005, 7(4): 419-423
[22] Ackermann J. Parameter space design of robust control systems[J]. IEEE Trans Automat Control, 1980, 25: 1058-1080
[23] Hinrichsen D, Pritchard A J. Stability radii of linear systems[J]. Systems Control Lett, 1986, 7: 1-10
[24] Hinrichsen D, Pritchard A J. A note on some differences between real and complex stability radii[J]. Systems Control Lett, 1990, 14: 401-408
[25] Su J H. Further results on the robust stability of linear systems with a single time delay[J]. Systems and Control Letters, 1994, 23: 375-379
[26] Tseng C L, Fong I K, Su J H. Robust stability analysis for uncertain delay systems with output feedback controller [J]. Systems and Control Letters, 1994, 23: 271-278
[27] Haussmann U G. Asymptotic stability of the linear Itô equation in infinite dimensions[M]. J Math Anal Appl, 1978, 65: 219-235
[28] Ichikawa A. Stability of semilinear stochastic evolution equations[J]. J Math Anal Appl, 1982, 90: 12-44
[29] Mao X, Koroleva N, Rodkina A. Robust stability of uncertain stochastic differential delay equations[J]. Systems and control letters, 1998, 35(5): 325-336
[30] Mao X. Exponential stability of stochastic delay interval systems with Markovian switching[J]. IEEE Trans Auto Control, 2002, 47(10): 1604-1612
[31] Hu L, Mao X, Zhang L. Robust stability and boundedness

of nonlinear hybrid stochastic differential delay equations [J]. IEEE Transactions on Automatic Control, 2013, 9: 2319-2332

[32] Arrowsmith D K, Place C M. An Introduction to Dynamical Systems [M]. Cambridge University Press, 1990

[33] Mao X. Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control [J]. Automatica, 2013, 49(12): 3677-3681

[34] Ji Y, Chizeck H J. Controllability, stabilizability and continuous-time Markovian jump linear quadratic control [J]. IEEE Trans Automat Control, 1990, 35: 777-788

[35] Mao X, Yin G, Yuan C. Stabilization and destabilization of hybrid systems of stochastic differential equations [J]. Automatica, 2007, 43: 264-273

[36] Wu L, Su X, Shi P. Sliding mode control with bounded L_2 gain performance of Markovian jump singular time-delay systems [J]. Automatica, 2012, 48(8): 1929-1933

[37] Mao X, Liu W, Hu L, et al. Stabilization of hybrid stochastic differential equations by feedback control based on discrete-time state observations [J]. Systems and Control Letters, 2014, 73: 88-95

[38] You S, Liu W, Lu J, et al. Stabilization of hybrid systems by feedback control based on discrete-time state observations [J]. SIAM J Control and Optimization, 2015, 53(2): 905-925

[39] Zhang Q. Stock trading: an optimal selling rule [J]. SIAM J Control Optim, 2001, 40(1): 64-87

[40] Mariton M. Jump linear systems in automatic control [M]. Marcel Dekker, 1990

[41] Song G, Zheng B C, Luo Q, et al. Stabilisation of hybrid stochastic differential equations by feedback control based on discrete-time observations of state and mode [J]. IET Control Theory & Applications, 2017, in press