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具反应扩散项的脉冲时滞 Cohen-Grossberg 神经网络的稳定性

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摘要

讨论了一类具反应扩散项的脉冲变时滞 Cohen-Grossberg 神经网络在 Neumann 边界条件下的稳定性. 通过构造分片连续的 Lyapunov 函数,结合 Neumann 特征值问题,并利用 Gronwall-Bellman 脉冲积分不等式,得到了新的与反应扩散项和时滞有关的保证平衡点全局指数稳定性的代数判据.

关键词

全局指数稳定性; Cohen-Grossberg 神经网络; 反应扩散; 变时滞; 脉冲

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0 引言

Cohen-Grossberg 神经网络(CGNNs)[1] 最早由 Cohen 和 Grossberg 于 1983 年提出,一直以来在众多研究领域中都有着广泛的应用,如优化计算、信号处理和图像传输等. 考虑到神经网络中硬件实现的开关滞后、参数的变化以及分布杂散参数释放特性的影响,许多学者进一步地提出了更具实际意义的时滞 Cohen-Grossberg 神经网络(DCGNNs). 无论 CGNNs 还是 DCGNNs,其相关的实际应用均依赖于自身的某些动力行为,如稳定性、收敛性、振动性等[2-6],特别地,稳定性是主要因素. CGNNs 和 DCGNNs 的稳定性具有重要的理论意义和实际意义,截至目前,已有大量研究工作并获得了许多稳定性判据[7-10].

另一方面,值得注意的是,在现实世界中,脉冲现象和扩散现象是普遍存在的,所以,具脉冲和反应扩散项的神经网络能够更精确地模拟事物的实际发展动态. 有鉴于此,具脉冲和反应扩散项的神经网络引起了众多学者的关注和兴趣,已取得许多有着重要意义的结论[11-26]. 文献[18]借助脉冲微分不等式考察了一类具时滞的脉冲反应扩散 CGNNs 的全局指数稳定性,获得了若干与反应扩散项无关的代数判据. 2010 年,文献[26]仍利用脉冲微分不等式进一步研究了具时滞的脉冲反应扩散 CGNNs 的全局指数稳定性. 与文献[18]不同的是,通过结合边界条件和利用扩散项的性质,文献[26]给出的稳定性充分条件与反应扩散项有关.

总结已有的关于脉冲微分系统稳定性的研究成果,可以发现所涉及的研究方法最终会归于脉冲微分不等式. 本文利用新的方法——Gronwall-Bellman 脉冲积分不等式,通过构造分片连续的 Lyapunov 函数,并结合 Neumann 特征值问题,研究了一类具反应扩散项的脉冲变时滞 Cohen-Grossberg 神经网络的稳定性,得到了新的简洁的保证平衡点全局指数稳定性的充分条件.

1 模型与定义

$\mathbf{R}^n$ 为 n 维 Euclidean 空间,$\Omega\subset\mathbf{R}^m$ 为具光滑边界 $\partial\Omega$ 的有界集,并且 $\mathrm{mes}\,\Omega>0$. $\mathbf{R}_+=[0,\infty)$,$t_0\in\mathbf{R}_+$. 考虑具反应扩散项的脉冲变时

滞 Cohen-Grossberg 神经网络:

$$\frac{\partial u_i(t,\boldsymbol{x})}{\partial t} = \sum_{s=1}^{m} \frac{\partial}{\partial x_s}\left(D_{is}\frac{\partial u_i(t,\boldsymbol{x})}{\partial x_s}\right) - a_i(u_i(t,\boldsymbol{x}))\left[\omega_i(u_i(t,\boldsymbol{x})) - \sum_{j=1}^{n} b_{ij}f_j(u_j(t,\boldsymbol{x})) - \sum_{j=1}^{n} c_{ij}f_j(u_j(t-\tau_j(t),\boldsymbol{x}))\right],$$

$$t \geqslant t_0,\quad t \neq t_k,\quad \boldsymbol{x} \in \boldsymbol{\Omega},\quad i = 1,2,\cdots,n,\quad k = 1,2,\cdots, \tag{1}$$

$$u_i(t_k+0,\boldsymbol{x}) = u_i(t_k,\boldsymbol{x}) + P_{ik}(u_i(t_k,\boldsymbol{x})),\quad \boldsymbol{x} \in \boldsymbol{\Omega},\quad k = 1,2,\cdots,\quad i = 1,2,\cdots,n. \tag{2}$$

其中: n 代表神经元的个数; $\boldsymbol{x} = (x_1,\cdots,x_m)^{\mathrm{T}} \in \boldsymbol{\Omega}$; $u_i(t,\boldsymbol{x})$ 为第 i 个神经元在时刻 t 和空间位置 $\boldsymbol{x}$ 的状态; 光滑函数 $D_{is} = D_{is}(t,\boldsymbol{x},\boldsymbol{u}) \geqslant 0$ 表示扩散算子; $a_i(u_i(t,\boldsymbol{x}))$ 为放大函数; $\omega_i(u_i(t,\boldsymbol{x}))$ 为行为函数; 激励函数 $f_j(u_j(t,\boldsymbol{x}))$ 代表第 j 个神经元在时刻 t 和空间位置 $\boldsymbol{x}$ 的输出; b_{ij},c_{ij} 为常数,用来刻画神经元间相互连接强度,即 b_{ij} 代表时刻 t 和空间位置 $\boldsymbol{x}$ 时第 j 个神经元作用于第 i 个神经元的强度,c_{ij} 代表时刻 $t-\tau_j(t)$ 和空间位置 $\boldsymbol{x}$ 时第 j 个神经元作用于第 i 个神经元的强度,其中 $\tau_j(t)$ 为轴突信号传输时滞,且满足 $0 \leqslant \tau_j(t) \leqslant \tau$ 和 $\dot{\tau}_j(t) < 1 - \frac{1}{h}(h>0)$; 时间序列 $\{t_k\}(k=1,2,\cdots)$ 为脉冲时刻序列,满足 $0 \leqslant t_0 < t_1 < t_2 < \cdots$ 和 $\lim_{k\to\infty} t_k = \infty$; 对应于固定的 $\boldsymbol{x}$,$u_i(t_k+0,\boldsymbol{x})$ 和 $u_i(t_k-0,\boldsymbol{x})$ 分别为 $u_i(t,\boldsymbol{x})$ 在时刻 t_k 的左极限和右极限; $P_{ik}(u_i(t_k,\boldsymbol{x}))$ 代表 $u_i(t,\boldsymbol{x})$ 在脉冲时刻 t_k 和空间位置 $\boldsymbol{x}$ 的突变.

记 $\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{u}(t,\boldsymbol{x};t_0,\boldsymbol{\varphi})$,$\boldsymbol{u} \in \mathbf{R}^n$ 为系统(1)—(2)满足初始条件

$$\boldsymbol{u}(s,\boldsymbol{x};t_0,\boldsymbol{\varphi}) = \boldsymbol{\varphi}(s,\boldsymbol{x}),\quad t_0 - \tau \leqslant s \leqslant t_0,\quad \boldsymbol{x} \in \boldsymbol{\Omega} \tag{3}$$

和 Neumann 边界条件

$$\frac{\partial \boldsymbol{u}(t,\boldsymbol{x};t_0,\boldsymbol{\varphi})}{\partial N} = 0,\quad t \geqslant t_0,\quad \boldsymbol{x} \in \partial\boldsymbol{\Omega} \tag{4}$$

的解,其中 $\frac{\partial}{\partial N}$ 为边界 $\partial\boldsymbol{\Omega}$ 上的单位外法向量,$\boldsymbol{\varphi}(s,\boldsymbol{x}) = (\varphi_1(s,\boldsymbol{x}),\cdots,\varphi_n(s,\boldsymbol{x}))^{\mathrm{T}}$,$\int_E \sum_{i=1}^{n} \boldsymbol{\varphi}_i^2(s,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ 在 $[t_0-\tau,t_0]$ 上有界,并且 $\varphi_i(s,\boldsymbol{x})(i=1,2,\cdots,n)$ 在 $[t_0-\tau,t_0]$ 上关于变量 s 一阶连续可导. 问题(1)—(4)的解 $\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{u}(t,\boldsymbol{x};t_0,\boldsymbol{\varphi}) = (u_1(t,\boldsymbol{x};t_0,\boldsymbol{\varphi}),\cdots,u_n(t,\boldsymbol{x};t_0,\boldsymbol{\varphi}))^{\mathrm{T}}$ 为关于变量 t 具有第一类间断点 $t_k(k=1,2,\cdots)$ 的分片连续函数,即

$$u_i(t_k-0,\boldsymbol{x}) = u_i(t_k,\boldsymbol{x}),$$
$$u_i(t_k+0,\boldsymbol{x}) = u_i(t_k,\boldsymbol{x}) + P_{ik}(u_i(t_k,\boldsymbol{x})).$$

本文中,定义 $\boldsymbol{u}(t,\boldsymbol{x};t_0,\boldsymbol{\varphi})$ 的模为

$$\|\boldsymbol{u}(t,\boldsymbol{x};t_0,\boldsymbol{\varphi})\|_{\boldsymbol{\Omega}} = \left(\sum_{i=1}^{n}\int_E u_i^2(t,\boldsymbol{x};t_0,\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}},$$

并假设:

(H1) 函数 $a_i(\cdot)$ 是正定的,连续的,并且有界的,即存在常数 $\underline{a}_i$ 和 $\overline{a}_i$ 使得

$$0 < \underline{a}_i \leqslant a_i(\zeta) \leqslant \overline{a}_i < \infty,\quad \zeta \in \mathbf{R},\quad i = 1,\cdots,n;$$

(H2) 函数 $\omega_i(\cdot)$ 是连续的,满足 $\omega_i(0) = 0$,并且存在常数 $p_i > 0$ 使得对于任意的 $\zeta_1,\zeta_2 \in \mathbf{R}$,$\zeta_1 \neq \zeta_2$,$i = 1,\cdots,n$,有

$$\frac{\omega_i(\zeta_1) - \omega_i(\zeta_2)}{\zeta_1 - \zeta_2} \geqslant p_i > 0;$$

(H3) 激励函数 $f_i(\cdot)$ 连续,满足 $f_i(0) = 0$,并且存在常数 $l_i > 0$ 使得对于任意的 $\zeta_1,\zeta_2 \in \mathbf{R}$,$\zeta_1 \neq \zeta_2$,$i = 1,\cdots,n$,有

$$l_i = \sup_{\zeta_1\neq\zeta_2}\frac{f_i(\zeta_1) - f_i(\zeta_2)}{\zeta_1 - \zeta_2};$$

(H4) 函数 $P_{ik}(\cdot)$ 在 $\mathbf{R}$ 上连续,且满足 $P_{ik}(0) = 0$,$i = 1,2,\cdots,n$,$k = 1,2,\cdots$.

根据假设(H1)—(H4),容易观察到系统(1)—(4)具有平衡点 $\boldsymbol{u} = 0$.

定义 1 系统(1)—(4)的平衡点 $\boldsymbol{u} = 0$ 称为全局指数稳定的,如果存在常数 $\kappa > 0$ 和 $M \geqslant 1$ 使得

$$\|\boldsymbol{u}(t,\boldsymbol{x};t_0,\boldsymbol{\varphi})\|_{\boldsymbol{\Omega}} \leqslant M\,\overline{\|\boldsymbol{\varphi}\|}_{\boldsymbol{\Omega}}\mathrm{e}^{-\kappa(t-t_0)},\quad t \geqslant t_0,$$

其中 $\overline{\|\boldsymbol{\varphi}\|}_{\boldsymbol{\Omega}}^2 = \sup_{t_0-\tau\leqslant s\leqslant t_0}\sum_{i=1}^{n}\int_E \boldsymbol{\varphi}_i^2(s,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$.

引理 1[27] (Gronwall-Bellman 脉冲积分不等式) 假设:

(A1) 时间序列 $\{t_k\}$ 满足 $0 \leqslant t_0 < t_1 < t_2 < \cdots$,且 $\lim_{k\to\infty} t_k = \infty$;

(A2) $q \in PC^1[\mathbf{R}_+,\mathbf{R}]$,且 $q(t)$ 在 t_k 为左连续,$k = 1,2,\cdots$;

(A3) $p \in C[\mathbf{R}_+,\mathbf{R}_+]$,且对于 $k = 1,2,\cdots$,有

$$q(t) \leqslant c + \int_{t_0}^{t} p(s)q(s)\,\mathrm{d}s + \sum_{t_0<t_k<t}\eta_k q(t_k),\quad t \geqslant t_0,$$

其中,$\eta_k \geqslant 0$,c 为常数,则

$$q(t) \leqslant c\prod_{t_0<t_k<t}(1+\eta_k)\exp\left(\int_{t_0}^{t} p(s)\,\mathrm{d}s\right),\quad t \geqslant t_0.$$

引理 2[28] (Poincaré 不等式). $S \subset \mathbf{R}^m$ 为具有光滑边界 ∂S 的有界区域,$v(\boldsymbol{x}) \in H^2(S) = \{v \mid v$ 的一阶及二阶导数 $L^2(S)$ 可积$\}$,且 $\left.\frac{\partial v(\boldsymbol{x})}{\partial N}\right|_{\partial S} = 0$,则

$$\int_S |v(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \leqslant \frac{1}{\lambda_1}\int_S |\nabla v(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x}.$$

其中 λ_1 为下列特征值问题的最小特征值:

$$-\Delta \Psi(\boldsymbol{x}) = \lambda \Psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \boldsymbol{S},$$

$$\frac{\partial \Psi(\boldsymbol{x})}{\partial N} = 0, \quad \boldsymbol{x} \in \partial \boldsymbol{S}.$$

引理 3 如果 $a>0, b>0$,则对于任意的 $\varepsilon>0$,不等式 $ab \leqslant \frac{1}{\varepsilon}a^2 + \varepsilon b^2$ 成立.

2 主要结论

定理 1 假设:

1) 存在常数 $\underline{D}>0$ 使得 $D_{is} = D_{is}(t,\boldsymbol{x},\boldsymbol{u}) \geqslant \underline{D}>0$,并记 $2\underline{D}\lambda_1 = \chi$;

2) $P_{ik}(u_i(t_k,\boldsymbol{x})) = -\theta_{ik}u_i(t_k,\boldsymbol{x}), 0 \leqslant \theta_{ik} \leqslant 2$;

3) 存在常数 γ 满足 $\gamma + \lambda + h\rho \mathrm{e}^{\gamma\tau} > 0$ 和 $\lambda + h\rho \mathrm{e}^{\gamma\tau} < 0$,其中

$$\lambda = \max_{i=1,\cdots,n}\left(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^n b_{ij}^2 + \overline{a}_i \sum_{j=1}^n c_{ij}^2\right) + \rho,$$

$$\rho = \max_{i=1,\cdots,n}(l_i^2) \sum_{i=1}^n \overline{a}_i,$$

则问题(1)—(4)的平衡点 $\boldsymbol{u}=0$ 是全局指数稳定的,且收敛速度为 $-\frac{\lambda + h\rho \mathrm{e}^{\gamma\tau}}{2}$.

证明 对式(1)两边同乘以 $u_i(t,\boldsymbol{x})$ 并在 Ω 上关于变量 $\boldsymbol{x}$ 积分,有

$$\frac{\mathrm{d}\left(\int_E u_i^2(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right)}{\mathrm{d}t} = 2\int_E \sum_{s=1}^m u_i(t,\boldsymbol{x}) \frac{\partial}{\partial x_s}\left(D_{is}\frac{\partial u_i(t,\boldsymbol{x})}{\partial x_s}\right)\mathrm{d}\boldsymbol{x} - 2\int_E u_i(t,\boldsymbol{x}) a_i(u_i(t,\boldsymbol{x}))\Big[\omega_i(u_i(t,\boldsymbol{x})) - \sum_{j=1}^n b_{ij}f_j(u_j(t,\boldsymbol{x})) - \sum_{j=1}^n c_{ij}f_j(u_j(t-\tau_j(t),\boldsymbol{x}))\Big]\mathrm{d}\boldsymbol{x},$$
$$t \geqslant t_0, \quad t \neq t_k, \quad k = 1,2,\cdots. \tag{5}$$

利用 Green 公式,结合 Neumann 边界条件,引理 2 和定理 1 的条件 1,有

$$2\sum_{s=1}^m \int_E u_i(t,\boldsymbol{x}) \frac{\partial}{\partial x_s}\left(D_{is}\frac{\partial u_i(t,\boldsymbol{x})}{\partial x_s}\right)\mathrm{d}\boldsymbol{x} = -2\sum_{s=1}^m \int_E D_{is}\left(\frac{\partial u_i(t,\boldsymbol{x})}{\partial x_s}\right)^2 \mathrm{d}\boldsymbol{x} \leqslant -2\underline{D}\lambda_1\int_E u_i^2(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} \triangleq -\chi\int_E u_i^2(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{6}$$

并且,根据(H1),(H2)和(H3),有如下估计:

$$2\int_E u_i(t,\boldsymbol{x}) a_i(u_i(t,\boldsymbol{x}))\omega_i(u_i(t,\boldsymbol{x}))\mathrm{d}\boldsymbol{x} \geqslant 2\underline{a}_i p_i \int_E |u_i(t,\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x}, \tag{7}$$

$$2\int_E u_i(t,\boldsymbol{x}) a_i(u_i(t,\boldsymbol{x})) \sum_{j=1}^n b_{ij}f_j(u_j(t,\boldsymbol{x}))\mathrm{d}\boldsymbol{x} \leqslant \overline{a}_i \sum_{j=1}^n \int_E (b_{ij}^2 u_i^2(t,\boldsymbol{x}) + f_j^2(u_j(t,\boldsymbol{x})))\mathrm{d}\boldsymbol{x} \leqslant \overline{a}_i \sum_{j=1}^n \int_E (b_{ij}^2 u_i^2(t,\boldsymbol{x}) + l_j^2 u_j^2(t,\boldsymbol{x}))\mathrm{d}\boldsymbol{x}, \tag{8}$$

$$2\int_E u_i(t,\boldsymbol{x}) a_i(u_i(t,\boldsymbol{x})) \sum_{j=1}^n c_{ij}f_j(u_j(t-\tau_j(t),\boldsymbol{x}))\mathrm{d}\boldsymbol{x} \leqslant \overline{a}_i \sum_{j=1}^n \int_E (c_{ij}^2 u_i^2(t,\boldsymbol{x}) + f_j^2(u_j(t-\tau_j(t),\boldsymbol{x})))\mathrm{d}\boldsymbol{x} \leqslant \overline{a}_i \sum_{j=1}^n \int_E (c_{ij}^2 u_i^2(t,\boldsymbol{x}) + l_j^2 u_j^2(t-\tau_j(t),\boldsymbol{x}))\mathrm{d}\boldsymbol{x}. \tag{9}$$

将式(6)—(9)代入式(5)有

$$\frac{\mathrm{d}\left(\int_E u_i^2(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right)}{\mathrm{d}t} \leqslant -\chi\int_E u_i^2(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} - 2\underline{a}_i p_i \int_E u_i^2(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \overline{a}_i \sum_{j=1}^n \int_E (b_{ij}^2 u_i^2(t,\boldsymbol{x}) + l_j^2 u_j^2(t,\boldsymbol{x}))\mathrm{d}\boldsymbol{x} + \overline{a}_i \sum_{j=1}^n \int_E (c_{ij}^2 u_i^2(t,\boldsymbol{x}) + l_j^2 u_j^2(t-\tau_j(t),\boldsymbol{x}))\mathrm{d}\boldsymbol{x},$$
$$t \geqslant t_0, \quad t \neq t_k, \quad k = 1,2,\cdots. \tag{10}$$

构造 Lyapunov 函数 $V_i(t) = \int_E u_i^2(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x}$. 显然,$V_i(t)$ 是具第一类间断点 $t_k(k=1,2,\cdots)$ 的分片连续函数,满足 $V_i(t_k-0) = V_i(t_k)(k=1,2,\cdots)$,并且,根据定理 1 的条件 2,有

$$u_i^2(t_k+0,\boldsymbol{x}) = (-\theta_{ik}u_i(t_k,\boldsymbol{x}) + u_i(t_k,\boldsymbol{x}))^2 = (1-\theta_{ik})^2 u_i^2(t_k,\boldsymbol{x}) \leqslant u_i^2(t_k,\boldsymbol{x})(k=1,2,\cdots),$$

所以

$$V_i(t_k+0) \leqslant V_i(t_k), \quad k = 0,1,2,\cdots. \tag{11}$$

令 $t \in (t_k, t_{k+1}), k = 0,1,2,\cdots$,沿着问题(1)—(4)的解计算 $\frac{\mathrm{d}V_i(t)}{\mathrm{d}t}$,有

$$\frac{\mathrm{d}V_i(t)}{\mathrm{d}t} \leqslant -\chi\int_E u_i^2(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} - 2\underline{a}_i p_i \int_E u_i^2(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \overline{a}_i \sum_{j=1}^n \int_E (b_{ij}^2 u_i^2(t,\boldsymbol{x}) + l_j^2 u_j^2(t,\boldsymbol{x}))\mathrm{d}\boldsymbol{x} + \overline{a}_i \sum_{j=1}^n \int_E (c_{ij}^2 u_i^2(t,\boldsymbol{x}) + l_j^2 u_j^2(t-\tau_j(t),\boldsymbol{x}))\mathrm{d}\boldsymbol{x} \leqslant$$
$$\left(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^n b_{ij}^2 + \overline{a}_i \sum_{j=1}^n c_{ij}^2\right)V_i(t) + \overline{a}_i \max_{i=1,\cdots,n}(l_i^2) \sum_{j=1}^n V_j(t) + \overline{a}_i \max_{i=1,\cdots,n}(l_i^2) \sum_{j=1}^n V_j(t-\tau_j(t)),$$

$$t \in (t_k, t_{k+1}), \quad k = 0,1,2,\cdots. \tag{12}$$

从而，对于函数 $V(t) = \sum_{i=1}^{n} V_i(t)$，有

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leqslant \Big(\max_{i=1,\cdots,n}\Big(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^{n} b_{ij}^2 + \overline{a}_i \sum_{j=1}^{n} c_{ij}^2\Big) + \max_{i=1,\cdots,n}(l_i^2) \sum_{i=1}^{n} \overline{a}_i\Big) V(t) + \max_{i=1,\cdots,n}(l_i^2) \sum_{i=1}^{n} \overline{a}_i \sum_{j=1}^{n} V_j(t-\tau_j) = \lambda V(t) + \rho \sum_{j=1}^{n} V_j(t-\tau_j(t)), t \in (t_k, t_{k+1}),$$
$$k = 0,1,2,\cdots, \tag{13}$$

其中

$$\max_{i=1,\cdots,n}(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^{n} b_{ij}^2 + \overline{a}_i \sum_{j=1}^{n} c_{ij}^2) + \max_{i=1,\cdots,n}(l_i^2) \sum_{i=1}^{n} \overline{a}_i = \lambda, \quad \max_{i=1,\cdots,n}(l_i^2) \sum_{i=1}^{n} \overline{a}_i = \rho.$$

构造 Lyapunov 函数 $V^*(t) = \mathrm{e}^{\gamma(t-t_0)} V(t)$，其中 γ 满足 $\gamma + \lambda + h\rho \mathrm{e}^{\gamma\tau} > 0$ 和 $\lambda + h\rho \mathrm{e}^{\gamma\tau} < 0$. 显然，$V^*(t)$ 也是具有第一类间断点 $t_k(k = 1,2,\cdots)$ 的分片连续函数，满足

$$V^*(t_k - 0) = V^*(t_k), \quad k = 1,2,\cdots.$$

和

$$V^*(t_k + 0) \leqslant V^*(t_k), \quad k = 0,1,2,\cdots. \tag{14}$$

令 $t \in (t_k, t_{k+1})$，$k = 0,1,2,\cdots$，由式(13)，有

$$\frac{\mathrm{d}V^*(t)}{\mathrm{d}t} = \gamma \mathrm{e}^{\gamma(t-t_0)} V(t) + \mathrm{e}^{\gamma(t-t_0)} \frac{\mathrm{d}V(t)}{\mathrm{d}t} \leqslant \gamma \mathrm{e}^{\gamma(t-t_0)} V(t) + (\lambda V(t) + \rho \sum_{j=1}^{n} V_j(t-\tau_j(t))) \mathrm{e}^{\gamma(t-t_0)} = (\gamma+\lambda) V^*(t) + \rho \mathrm{e}^{\gamma(t-t_0)} \sum_{j=1}^{n} V_j(t-\tau_j(t)),$$
$$t \in (t_k, t_{k+1}), \quad k = 0,1,2,\cdots. \tag{15}$$

取足够小的 $\varepsilon > 0$，对(15)关于变量 t 从 $t_k + \varepsilon$ 至 t 积分有

$$V^*(t) \leqslant V^*(t_k + \varepsilon) + (\gamma+\lambda) \int_{t_k+\varepsilon}^{t} V^*(s)\mathrm{d}s + \int_{t_k+\varepsilon}^{t} \rho \mathrm{e}^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\mathrm{d}s,$$
$$t \in (t_k, t_{k+1}), \quad k = 0,1,2,\cdots, \tag{16}$$

式(16)中令 $\varepsilon \to 0$ 有

$$V^*(t) \leqslant V^*(t_k + 0) + (\gamma+\lambda) \int_{t_k}^{t} V^*(s)\mathrm{d}s + \int_{t_k}^{t} \rho \mathrm{e}^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\mathrm{d}s,$$
$$t \in (t_k, t_{k+1}), \quad k = 0,1,2,\cdots, \tag{17}$$

取 $t = t_{k+1} - \varepsilon$，$\varepsilon$ 为足够小的正数. 由(17)，有

$$V^*(t_{k+1} - \varepsilon) \leqslant V^*(t_k + 0) + (\gamma+\lambda) \int_{t_k}^{t_{k+1}-\varepsilon} V^*(s)\mathrm{d}s + \int_{t_k}^{t_{k+1}-\varepsilon} \rho \mathrm{e}^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\mathrm{d}s,$$
$$k = 0,1,2,\cdots, \tag{18}$$

从而

$$V^*(t_{k+1} - 0) \leqslant V^*(t_k + 0) + (\gamma+\lambda) \int_{t_k}^{t_{k+1}} V^*(s)\mathrm{d}s + \int_{t_k}^{t_{k+1}} \rho \mathrm{e}^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\mathrm{d}s,$$
$$k = 0,1,2,\cdots.$$

因为 $V^*(t_{k+1} - 0) = V^*(t_{k+1})$，$k = 0,1,2,\cdots$，所以，对于 $k = 0,1,2,\cdots$，

$$V^*(t_{k+1}) \leqslant V^*(t_k + 0) + (\gamma+\lambda) \int_{t_k}^{t_{k+1}} V^*(s)\mathrm{d}s + \int_{t_k}^{t_{k+1}} \rho \mathrm{e}^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\mathrm{d}s. \tag{19}$$

综合(17)和(19)，有

$$V^*(t) \leqslant V^*(t_k + 0) + (\gamma+\lambda) \int_{t_k}^{t} V^*(s)\mathrm{d}s + \int_{t_k}^{t} \rho \mathrm{e}^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\mathrm{d}s,$$
$$t \in (t_k, t_{k+1}], \quad k = 0,1,2,\cdots. \tag{20}$$

结合式(14)，可以得到，对于 $t \in (t_k, t_{k+1}]$，$k = 0,1,2,\cdots$，

$$V^*(t) \leqslant V^*(t_k) + (\gamma+\lambda) \int_{t_k}^{t} V^*(s)\mathrm{d}s + \int_{t_k}^{t} \rho \mathrm{e}^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\mathrm{d}s. \tag{21}$$

利用假设 $0 \leqslant \tau_j(t) \leqslant \tau$ 和 $\dot{\tau}_j(t) < 1 - \frac{1}{h}(h > 0)$，有

$$\int_{t_k}^{t} \rho \mathrm{e}^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\mathrm{d}s = \sum_{j=1}^{n} \int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} \rho \mathrm{e}^{\gamma(\theta+\tau_j(s)-t_0)} V_j(\theta) \frac{1}{1-\dot{\tau}_j(s)}\mathrm{d}\theta \leqslant h\rho \mathrm{e}^{\gamma\tau} \sum_{j=1}^{n} \int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} \mathrm{e}^{\gamma(\theta-t_0)} V_j(\theta)\mathrm{d}\theta,$$

所以

$$V^*(t) \leqslant V^*(t_k) + (\gamma+\lambda) \int_{t_k}^{t} V^*(s)\mathrm{d}s + h\rho \mathrm{e}^{\gamma\tau} \sum_{j=1}^{n} \int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} \mathrm{e}^{\gamma(s-t_0)} V_j(s)\mathrm{d}s,$$
$$t \in (t_k, t_{k+1}], \quad k = 0,1,2,\cdots,$$

从而

$$V^*(t_k) \leqslant V^*(t_{k-1}) + (\gamma+\lambda) \int_{t_{k-1}}^{t_k} V^*(s)\mathrm{d}s +$$

$$h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_{k-1}-\tau_j(t_{k-1})}^{t_k-\tau_j(t_k)} e^{\gamma(s-t_0)}V_j(s)\,ds,$$
$$\vdots$$
$$V^*(t_2)\leqslant V^*(t_1)+(\gamma+\lambda)\int_{t_1}^{t_2}V^*(s)\,ds+$$
$$h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_1-\tau_j(t_1)}^{t_2-\tau_j(t_2)} e^{\gamma(s-t_0)}V_j(s)\,ds,$$
$$V^*(t_1)\leqslant V^*(t_0)+(\gamma+\lambda)\int_{t_0}^{t_1}V^*(s)\,ds+$$
$$h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_1-\tau_j(t_1)} e^{\gamma(s-t_0)}V_j(s)\,ds,$$

故

$$V^*(t)\leqslant V^*(t_0)+(\gamma+\lambda)\int_{t_0}^{t}V^*(s)\,ds+$$
$$h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t-\tau_j(t)} e^{\gamma(s-t_0)}V_j(s)\,ds\leqslant$$
$$V^*(t_0)+(\gamma+\lambda+h\rho e^{\gamma\tau})\int_{t_0}^{t}V^*(s)\,ds+$$
$$h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(s-t_0)}V_j(s)\,ds,$$
$$t\in(t_k,t_{k+1}],\quad k=0,1,2,\cdots.$$

因为

$$h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(s-t_0)}V_j(s)\,ds\leqslant$$
$$h\rho e^{\gamma\tau}\int_{t_0-\tau}^{t_0}\Big(\sum_{j=1}^{n}\int_E\varphi_j^2(s,\boldsymbol{x})\,d\boldsymbol{x}\Big)ds\leqslant$$
$$\tau h\rho e^{\gamma\tau}\ \overline{\|\boldsymbol{\varphi}\|}_{\Omega}^{2},$$

所以

$$V^*(t)\leqslant V^*(t_0)+\tau h\rho e^{\gamma\tau}\ \overline{\|\boldsymbol{\varphi}\|}_{\Omega}^{2}+$$
$$(\gamma+\lambda+h\rho e^{\gamma\tau})\int_{t_0}^{t}V^*(s)\,ds,$$
$$t\in(t_k,t_{k+1}],\quad k=0,1,2,\cdots.$$

根据引理1，得到

$$V^*(t)\leqslant\big(V^*(t_0)+\tau h\rho e^{\gamma\tau}\ \overline{\|\boldsymbol{\varphi}\|}_{\Omega}^{2}\big)\cdot$$
$$\exp\{(\gamma+\lambda+h\rho e^{\gamma\tau})(t-t_0)\},\quad t\geqslant t_0,$$

即

$$\|\boldsymbol{u}(t,\boldsymbol{x};t_0,\boldsymbol{\varphi})\|_{\Omega}\leqslant\sqrt{1+\tau h\rho e^{\gamma\tau}}\ \overline{\|\boldsymbol{\varphi}\|}_{\Omega}\cdot$$
$$\exp\Big\{\Big(\frac{\lambda+h\rho e^{\gamma\tau}}{2}\Big)(t-t_0)\Big\},\quad t\geqslant t_0.$$

证毕.

定理2 假设：

1) 存在常数 $\underline{D}>0$ 使得 $D_{is}=D_{is}(t,\boldsymbol{x},\boldsymbol{u})\geqslant\underline{D}>0$，并记 $2\underline{D}\lambda_1=\chi$；

2) $P_{ik}(u_i(t_k,\boldsymbol{x}))=-\theta_{ik}u_i(t_k,\boldsymbol{x})$，$0\leqslant\theta_{ik}\leqslant 2$；

3) 存在常数 γ 和 $\varepsilon_1,\varepsilon_2>0$ 使得 $\gamma+\lambda+h\rho e^{\gamma\tau}>0$，$\lambda+h\rho e^{\gamma\tau}<0$，其中

$$\rho=\frac{\max\limits_{i=1,\cdots,n}(l_i^2)}{\varepsilon_2}\sum_{i=1}^{n}\bar{a}_i,\ \lambda=$$
$$\max_{i=1,\cdots,n}\Big(-\chi-2\underline{a}_ip_i+\bar{a}_i\sum_{j=1}^{n}(\varepsilon_1b_{ij}^2+\varepsilon_2c_{ij}^2)\Big)+\frac{\max\limits_{i=1,\cdots,n}(l_i^2)}{\varepsilon_1}\sum_{i=1}^{n}\bar{a}_i,$$

则问题(1)—(4)的平衡点 $\boldsymbol{u}=0$ 是全局指数稳定的.

证明 由引理3，有

$$2\sum_{j=1}^{n}b_{ij}\int_E u_i(t,\boldsymbol{x})f(u_j(t,\boldsymbol{x}))\,d\boldsymbol{x}\leqslant$$
$$\sum_{j=1}^{n}\int_E\Big(\varepsilon_1b_{ij}^2u_i^2(t,\boldsymbol{x})+\frac{l_j^2}{\varepsilon_1}u_j^2(t,\boldsymbol{x})\Big)d\boldsymbol{x}$$

和

$$2\sum_{j=1}^{n}c_{ij}\int_E u_i(t,\boldsymbol{x})f(u_j(t-\tau_j,\boldsymbol{x}))\,d\boldsymbol{x}\leqslant$$
$$\sum_{j=1}^{n}\int_E\Big(\varepsilon_2c_{ij}^2u_i^2(t,\boldsymbol{x})+\frac{l_j^2}{\varepsilon_2}u_j^2(t-\tau_j,\boldsymbol{x})\Big)d\boldsymbol{x}.$$

类似于定理1，可证明定理2，这里省略.

3 实例

考虑下列具反应扩散项的脉冲变时滞CCNN

$$\frac{\partial u_i(t,\boldsymbol{x})}{\partial t}=\sum_{s=1}^{m}\frac{\partial}{\partial x_s}\Big(D_{is}\frac{\partial u_i(t,\boldsymbol{x})}{\partial x_s}\Big)-$$
$$a_i(u_i(t,\boldsymbol{x}))\Big[\omega_i(u_i(t,\boldsymbol{x}))-\sum_{j=1}^{n}b_{ij}f_j(u_j(t,\boldsymbol{x}))-$$
$$\sum_{j=1}^{n}c_{ij}f_j(u_j(t-\tau_j(t),\boldsymbol{x}))\Big],$$
$$t\geqslant 0,\quad t\neq t_k,\quad \boldsymbol{x}\in\boldsymbol{\Omega},\quad k=1,2,\cdots,$$
$$u_1(t_k+0,\boldsymbol{x})=u_1(t_k,\boldsymbol{x})+1.343u_1(t_k,\boldsymbol{x}),$$
$$u_2(t_k+0,\boldsymbol{x})=u_2(t_k,\boldsymbol{x})+1.343u_2(t_k,\boldsymbol{x}),$$
$$k=1,2,\cdots,\quad \boldsymbol{x}\in\boldsymbol{\Omega}.$$

其中，初边值条件分别为(3)和(4)，$n=2$，$m=2$，

$$\Omega=\{(x_1,x_2)^{\mathrm T}\mid 0<|x_1|<\pi,0<|x_2|<2\},$$
$$a_1(u_1(t,\boldsymbol{x}))=a_2(u_2(t,\boldsymbol{x}))=1,$$
$$\omega_1(u_1(t,\boldsymbol{x}))=6.5u_1(t,\boldsymbol{x}),$$
$$\omega_2(u_2(t,\boldsymbol{x}))=8.5u_2(t,\boldsymbol{x}),$$
$$(D_{is})_{2\times 2}=\begin{pmatrix}1.2 & 2.3\\ 2.2 & 1.5\end{pmatrix},$$
$$(b_{ij})_{2\times 2}=\begin{pmatrix}-0.23 & 1.3\\ -0.14 & 3.2\end{pmatrix},$$
$$(c_{ij})_{2\times 2}=\begin{pmatrix}-0.1 & -0.2\\ 0.25 & -0.13\end{pmatrix},$$
$$f_j(u_j)=\frac{\sqrt{2}}{4}(|u_j+1|-|u_j-1|)\ (j=1,2),$$

$0 \leqslant \tau_j(t) \leqslant 0.5(j = 1,2)$,

$\dot{\tau}_j(t) < 0(j = 1,2)$.

由于 $\lambda_1 = 1, \underline{D} = 1.2$,计算有 $\chi = 2.4$. 取 $l_i = \frac{\sqrt{2}}{2}, \underline{a}_i = \overline{a}_i = 1, p_i = 6.5(i = 1,2)$,则

$$\rho = \max_{i=1,\cdots,n}(l_i^2)\sum_{i=1}^{n}\overline{a}_i = 1,$$

$$\lambda = \max_{i=1,\cdots,n}\left(-\chi - 2\underline{a}_i p_i + \overline{a}_i\sum_{j=1}^{n}b_{ij}^2 + \overline{a}_i\sum_{j=1}^{n}c_{ij}^2\right) + \rho = -4.061.$$

选取 $\gamma = 2.6, \tau = 0.5, h = 1$,有

$\gamma + \lambda + h\rho e^{\gamma\tau} = 2.6 - 4.061 + e^{1.3} > 0$,

$\lambda + h\rho e^{\gamma\tau} = -4.061 + e^{1.3} < 0$.

根据定理 1,可以知道该系统的平衡点 $u = 0$ 为全局指数稳定.

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Stability analysis for impulsive reaction-diffusion Cohen-Grossberg neural networks with time-varying delays

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Abstract This work concerns the stability for a class of impulsive Cohen-Grossberg neural networks with time-varying delays，reaction-diffusion and Neumann boundary condition. By means of the impulsive integral inequality of Gronwall-Bellman type and piecewise continuous Lyapunov functions as well as Neumann eigenvalue problem，we summarize some new and concise sufficient conditions ensuring the global exponential stability of the equilibrium point. Moreover，the provided stability criteria are shown to be associated with both reaction-diffusion and time delays. An illustrative example is finally given to demonstrate the effectiveness of our obtained results.

Key words global exponential stability; Cohen-Grossberg neural network; reaction-diffusion; time-varying delay; impulse