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高斯过程下的部分和不等式

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摘要

主要讨论高斯平稳过程增量的一个结果.综合考虑高斯过程在区间[0,h]上一个线性组合后,重新改写了尾概率不等式,推广了现有结果.

关键词

高斯过程;增量;相互独立;单调非降;线性组合

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0 引言

要证明高斯过程的连续模和讨论大增量问题[1-6],先要证明一个十分重要的引理,它是关于高斯过程增量的尾概率估计. 高斯过程增量的一系列不等式都由此推出. 这里引进线性组合的概念,重新改写尾概率不等式,使之适用范围更广. 其中当 $d=1$ 时,就得到类似引理的结果. 高斯过程 $\{\Gamma(t),-\infty<t<\infty\}$ 的概念可参见文献[7]. 现给出如下引理.

引理 令 $\{\Gamma(t),-\infty<t<\infty\}$ 是几乎处处连续的高斯过程且 $E\Gamma(t)=0$ 且

$$E(\Gamma(t+s)-\Gamma(t))^2=\sigma^2(s). \tag{1}$$

假设 $\sigma^2(s)$ 关于 s 是单调非降的,则

$$P\left\{\sup_{0\leqslant t\leqslant T-a}\sup_{0\leqslant s\leqslant a}|\Gamma(t+s)-\Gamma(t)|\geqslant u\sigma\left(a+\frac{1}{R}\right)+2\sum_{j=0}^{\infty}x_j\sigma\left(\frac{1}{2^{r+j+1}}\right)\right\}\leqslant 2TR(Ra+1)\mathrm{e}^{-\frac{u^2}{2}}+8TR\sum_{j=0}^{\infty}2^j\mathrm{e}^{-\frac{x_j^2}{2}} \tag{2}$$

对于任何正实数 T,a,x_j 和整数 r 都成立,这里令 $R=2^r$.

证明参见文献[8].

1 主要结果

定理 设 d 是正整数,$a_1,a_2,\cdots,a_d$ 是实数,满足 $\sum_{i=1}^{d}a_i^2=1$. 记

$$S_d=\sum_{i=1}^{d}a_i\left[\Gamma\left(t+\frac{i}{d}s\right)-\Gamma\left(t+\frac{i-1}{d}s\right)\right], \tag{3}$$

则有

$$P\left\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}|S_d|\geqslant u\sigma\left(\frac{h}{d}+\frac{1}{R}\right)+2\sum_{j=0}^{\infty}x_j\sigma\left(\frac{1}{2^{r+j+1}}\right)\right\}\leqslant 2R\left(\frac{Rh}{d}+1\right)^d\mathrm{e}^{-\frac{u^2}{2}}+8R\sum_{j=1}^{\infty}2^j\mathrm{e}^{-\frac{x_j^2}{2}} \tag{4}$$

对于任何正实数 h,x_j 和整数 r 都成立,这里令 $R=2^r$.

证明 对正实数 $t\leqslant 1-h$ 和正整数 r, 记

$$t_r=[2^rt]/2^r, \tag{5}$$

记号[·]表示取最大整数部分.

对每一 $\omega\in\Omega$ 和固定的 t,s,r 有

$$|S_d| = \left|\sum_{i=1}^{d} a_i\Gamma\left(t+\frac{i}{d}s\right) - \sum_{i=1}^{d} a_i\Gamma\left(t+\frac{i-1}{d}s\right)\right| =$$

$$\left|\sum_{i=1}^{d} a_i\Gamma\left(t+\frac{i}{d}s\right) + \sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i}{d}s\right)_r\right) - \sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i}{d}s\right)_r\right) + \sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i-1}{d}s\right)_r\right) - \sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i-1}{d}s\right)_r\right) - \sum_{i=1}^{d} a_i\Gamma\left(t+\frac{i-1}{d}s\right)\right| \leqslant$$

$$\left|\sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i}{d}s\right)_r\right) - \sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i-1}{d}s\right)_r\right)\right| + \sum_{j=0}^{\infty}\left|\sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i}{d}s\right)_{r+j+1}\right) - \sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i}{d}s\right)_{r+j}\right)\right| + \sum_{j=0}^{\infty}\left|\sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i-1}{d}s\right)_{r+j+1}\right) - \sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i-1}{d}s\right)_{r+j}\right)\right| \equiv U_1+U_2+U_3. \quad (6)$$

由于

$$\sup_{0<s\leqslant h}\left|\left(t+\frac{i}{d}s\right)_r - \left(t+\frac{i-1}{d}s\right)_r\right| = \sup_{0<s\leqslant h}\left|\frac{\left[\left(t+\frac{i}{d}s\right)2^r\right]}{2^r} - \frac{\left[\left(t+\frac{i-1}{d}s\right)2^r\right]}{2^r}\right| \leqslant \sup_{0<s\leqslant h}\left|\frac{\left(t+\frac{i}{d}s\right)2^r - \left(t+\frac{i-1}{d}s\right)2^r + 1}{2^r}\right| \leqslant \sup_{0<s\leqslant h}\left|\frac{\frac{s2^r}{d}+1}{2^r}\right| \leqslant \frac{h}{d}+R^{-1} \quad (7)$$

对 $\forall i=1,2,\cdots,d$ 都成立. 并有

$$\sup_{0<s\leqslant h}\left|\left(t+\frac{i}{d}s\right)_{r+j+1} - \left(t+\frac{i}{d}s\right)_{r+j}\right| \leqslant 2^{-(r+j+1)}, \quad (8)$$

且

$$\Gamma\left(\left(t+\frac{i}{d}s\right)_r\right) - \Gamma\left(\left(t+\frac{i-1}{d}s\right)_r\right) \sim N\left(0, \sigma^2\left(\left(t+\frac{i}{d}s\right)_r - \left(t+\frac{i-1}{d}s\right)_r\right)\right). \quad (9)$$

对任意的正数 h, u, x_j 和整数 r, j，令

$$K_{i-1} = \left[\left(t+\frac{i-1}{d}s\right)2^r\right], \quad L_i = K_i - K_{i-1},$$

$$E\left(\Gamma\left(\frac{K_{i-1}}{R}+\frac{L_i}{R}\right) - \Gamma\left(\frac{K_{i-1}}{R}\right)\right)^2 = \sigma^2\left(\frac{L_i}{R}\right) \equiv \sigma_i,$$

则有 $\sigma_i \leqslant \sigma^2\left(\frac{h}{d}+R^{-1}\right), \forall i=1,2,\cdots,d$.

因而

$$E\left(\sum_{i=1}^{d} a_i\Gamma\left(\frac{K_{i-1}}{R}+\frac{L_i}{R}\right) - \sum_{i=1}^{d} a_i\Gamma\left(\frac{K_{i-1}}{R}\right)\right)^2 = \sum_{i=1}^{d} a_i^2\sigma_i \leqslant \sum_{i=1}^{d} a_i^2\sigma^2\left(\frac{h}{d}+R^{-1}\right) = \sigma^2\left(\frac{h}{d}+R^{-1}\right).$$

再由正态分布的尾概率不等式[9]，有

$$P\left\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}\left|\sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i}{d}s\right)_r\right) - \sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i-1}{d}s\right)_r\right)\right| \geqslant u\sigma\left(\frac{h}{d}+\frac{1}{R}\right)\right\} \leqslant$$

$$P\left\{\max_{0\leqslant K_{i-1}\leqslant\left[\left(1-\frac{d+1-i}{d}h\right)R\right]}\max_{0\leqslant L_i\leqslant\left[\frac{Rh}{d}+1\right]}\left|\sum_{i=1}^{d} a_i\Gamma\left(\frac{K_{i-1}}{R}+\frac{L_i}{R}\right) - \sum_{i=1}^{d} a_i\Gamma\left(\frac{K_{i-1}}{R}\right)\right| \geqslant u\sigma\left(\frac{h}{d}+\frac{1}{R}\right)\right\} \leqslant$$

$$2R\left(\frac{Rh}{d}+1\right)^d e^{-\frac{u^2}{2}}, \quad (10)$$

$$P\left\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0<s\leqslant h}\left|\sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i}{d}s\right)_{r+j+1}\right) - \sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i}{d}s\right)_{r+j}\right)\right| \geqslant x_j\sigma\left(\frac{1}{2^{r+j+1}}\right)\right\} \leqslant 2\cdot 2^{r+j+1}\cdot e^{-\frac{x_j^2}{2}}, \quad (11)$$

$$P\left\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0<s\leqslant h}\left|\sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i-1}{d}s\right)_{r+j+1}\right) - \sum_{i=1}^{d} a_i\Gamma\left(\left(t+\frac{i-1}{d}s\right)_{r+j}\right)\right| \geqslant x_j\sigma\left(\frac{1}{2^{r+j+1}}\right)\right\} \leqslant 2\cdot 2^{r+j+1}\cdot e^{-\frac{x_j^2}{2}}. \quad (12)$$

从而

$$P\left\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}|S_d| \geqslant u\sigma\left(\frac{h}{d}+\frac{1}{R}\right) + 2\sum_{j=0}^{\infty} x_j\sigma\left(\frac{1}{2^{r+j+1}}\right)\right\} \leqslant$$

$$P\left\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}U_1 + \sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}U_2 + \sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}U_3 \geqslant u\sigma\left(\frac{h}{d}+\frac{1}{R}\right) + 2\sum_{j=0}^{\infty} x_j\sigma\left(\frac{1}{2^{r+j+1}}\right)\right\} \leqslant$$

$$2R\left(\frac{Rh}{d}+1\right)^d e^{-\frac{u^2}{2}} + 2\sum_{j=0}^{\infty} 2\cdot 2^{r+j+1}\cdot e^{-\frac{x_j^2}{2}} = 2R\left(\frac{Rh}{d}+1\right)^d e^{-\frac{u^2}{2}} + 8R\sum_{j=0}^{\infty} 2^j\cdot e^{-\frac{x_j^2}{2}}.$$

证毕.

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Partial sum inequality in Gaussian process

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Abstract The paper mainly studies into a result of stationary increment in Gaussian process. Considering a linear combination of partial sum on interval $[0,h]$ comperhensively,this paper rewrites the inequality of tail probability and extends the existing result.

Key words Gaussian process;increment;mutually independent;monotone non-decreasing;linear combination