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# Nonlinear boundary stabilization of a compactly coupled system of wave equations with variable coefficients

LEI Juan[1]

**Abstract**

In this paper, we investigate the stabilization of the compactly coupled wave equations with variable coefficients and nonlinear dissipative boundary feedback. The energy decay estimates of solutions are obtained by the Riemannian geometry method and the multiplier technique.

**Key words**

multiplier method; nonlinear boundary feedback; riemannian manifold; energy decay.

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**Biography**

LEI Juan, female, master, mainly research on partial differential equations. leijuan 306@126.com

1 School of Mathematics and Physics, Nanjing University of Information Science and Technology, Nanjing 210044, China

## 0 Introduction

In this paper, we investigate the following evolutionary system

$$\begin{cases} u_{tt} + \mathscr{A}u + \alpha(u-v) = 0, & \text{in } \Omega \times \mathbf{R}_+, \\ v_{tt} + \mathscr{A}v + \alpha(v-u) = 0, & \text{in } \Omega \times \mathbf{R}_+, \\ u = v = 0, & \text{on } \Gamma_0 \times \mathbf{R}_+, \\ \dfrac{\partial u}{\partial \nu_{\mathscr{A}}} + a_1 u + g_1(u_t) = 0, & \text{on } \Gamma_1 \times \mathbf{R}_+, \\ \dfrac{\partial v}{\partial \nu_{\mathscr{A}}} + a_2 v + g_2(v_t) = 0, & \text{on } \Gamma_1 \times \mathbf{R}_+, \\ u(0) = u_0, u_t(0) = u_1, & \text{in } \Omega, \\ v(0) = v_0, v_t(0) = v_1, & \text{in } \Omega. \end{cases} \tag{1}$$

where $\Omega$ is a bounded domain in $\mathbf{R}^n$ $(n \geqslant 1)$ with smooth boundary $\Gamma$ which consists of two parts: $\Gamma_0$ and $\Gamma_1$, $\Gamma_0 \cup \Gamma_1 = \Gamma$, with $\Gamma_1$ nonempty, $\mathscr{A}$ is the operator defined by

$$\mathscr{A}u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(\boldsymbol{x}) \frac{\partial u}{\partial x_j}\right). \tag{2}$$

where $a_{ij} = a_{ji}$ are $C^\infty$ functions in $\mathbf{R}^n$ for $1 \leqslant i,j \leqslant n$ and

$$\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x})\xi_i\xi_j > 0, \quad \boldsymbol{x} \in \mathbf{R}^n, \quad \boldsymbol{\xi} = (\xi_1,\xi_2,\cdots,\xi_n) \neq 0, \quad \boldsymbol{\xi} \in \mathbf{R}^n, \tag{3}$$

$\alpha: \mathbf{R} \to \mathbf{R}$, $a_1, a_2: \Gamma_1 \to \mathbf{R}$, $g_1, g_2: \mathbf{R} \to \mathbf{R}$ are some given functions, $\dfrac{\partial u}{\partial \nu_{\mathscr{A}}} = \sum_{i,j=1}^{n} a_{ij} \dfrac{\partial u}{\partial x_j} \nu_i$ is the co-normal derivative and $\boldsymbol{\nu} = (\nu_1, \nu_2, \cdots, \nu_n)$ is the unit outward normal on $\Gamma$.

The problems of observation, control and stabilization for the wave equations have been widely studied. In the case where the coefficients $a_{ij}$ are constants (i.e., $\mathscr{A} = -\Delta$), energy decay rates were obtained by [1-6] and many other papers. For the case of variable coefficients, the exact controllability problems have been studied by several authors. For example, Yao[7] derived the geometrical conditions for exact controllability of wave equation with variable coefficients principal part. The main tool of [7] is the geometrical method which is useful in generalizing some results obtained in constant coefficient case to variable coefficient case. Feng[8] extended the results of Zuazua[6] to the variable coefficients case by using Riemannian geometry method and the integral inequality introduced in [3]. Recently, Guo and Shao[9] considered the uniform stabilization of an originally regarded non-dissipative system described by a semilinear wave equation with variable coefficients under the nonlinear boundary feedback. By introducing an equivalent energy function and using the energy multiplier method on the Riemannian manifold, they obtained the exponential stability of the system. Moreover, Yao[10] has discussed decay of the energy for the Cauchy problem of the wave equation of variable coefficients with a dissipation. We refer the readers to [11-13] for recent contributions in this direction.

In the case of coupled system of wave equations we can mention [14-17], in most of which the case of constant coefficients is considered. For example, Aassila[14] studied the decay property of the solutions to the evolutionary system (1) with $\mathscr{A} = -\Delta$ and obtained decay estimates of the energy of solutions. However, very little is known of the variable coefficients case. The purpose of this paper is to extend the result of Aassila[14] to the variable coefficients case. The main tools are the Riemannian geometry method and the integral inequality technique which are powerful to cope with variable coefficients. These methods are used in [8] for the wave equation with variable coefficients and will be extended here for the compactly coupled wave equations.

Our paper is organized as follows. In section 1, we list our main result. In section 2, we give the proof of the main result: Theorem 2.

## 1 Preliminaries and main result

To begin with, we introduce some notations and refer the reader to Yao[7] for further understanding of these notations. Moreover, we will apply some results in [7] to proof of our main theorem.

Let $\boldsymbol{A}(\boldsymbol{x}) = (a_{ij}(\boldsymbol{x}))$ for $\boldsymbol{x} \in \mathbf{R}^n$ be an $n \times n$ matrix, where $a_{ij}$ are the same as that in (2). Let $\mathbf{R}^n$ have the usual topology and $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ be the natural coordinate system. Denote $G(\boldsymbol{x}) = (g_{ij}(\boldsymbol{x})) = \boldsymbol{A}(\boldsymbol{x})^{-1}$, $\forall \boldsymbol{x} \in \mathbf{R}^n$. For each $\boldsymbol{x} \in \mathbf{R}^n$ define the inner product and the norm on the tangent space $\mathbf{R}^n_x = \mathbf{R}^n$ by

$$g(\boldsymbol{X}, \boldsymbol{Y}) = \langle \boldsymbol{X}, \boldsymbol{Y} \rangle_g = \sum_{i,j=1}^{n} g_{ij}(\boldsymbol{x}) \alpha_i \beta_j,$$
$$| \boldsymbol{X} |_g = \langle \boldsymbol{X}, \boldsymbol{X} \rangle_g^{\frac{1}{2}},$$
$$\forall \boldsymbol{X} = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i},$$
$$\boldsymbol{Y} = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \in \mathbf{R}^n_x.$$

$(\mathbf{R}^n, g)$ is a Riemannian manifold with Riemannian metric $g$. Denote the gradient of u in Riemannian manifold $(\mathbf{R}^n, g)$ by $\nabla_g u$, then we have

$$\nabla_g u = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \right) \frac{\partial}{\partial x_i},$$
$$| \nabla_g u |_g^2 = \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

Denote the Levi-Civita connection by $D$ on $(\mathbf{R}^n, g)$. Let $\boldsymbol{H}$ be a vector field on $(\mathbf{R}^n, g)$, the covariant differential $D\boldsymbol{H}$ of $\boldsymbol{H}$ determines a bilinear form on $\mathbf{R}^n_x \times \mathbf{R}^n_x$ for each $\boldsymbol{x} \in \mathbf{R}^n$ by

$$D\boldsymbol{H}(\boldsymbol{X}, \boldsymbol{Y}) = \langle \boldsymbol{D_X H}, \boldsymbol{Y} \rangle_g, \quad \forall \boldsymbol{X}, \boldsymbol{Y} \in \mathbf{R}^n_x.$$

where $\boldsymbol{D_X H}$ is the covariant derivative of vector field $\boldsymbol{H}$ with respect to $\boldsymbol{X}$. We set $H^1_{\Gamma_0}(\Omega) = \{ u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_0 \}$, where $H^k(\Omega)$ is the usual Sobolev space of order $k$.

The energy $E(t)$ of system (2) is definde by:

$$E(t) = \frac{1}{2} \int_\Omega (u_t^2 + v_t^2 + | \nabla_g u |_g^2 + | \nabla_g v |_g^2) \mathrm{d}x + \frac{\alpha}{2} \int_\Omega (u - v)^2 \mathrm{d}x + \frac{1}{2} \int_{\Gamma_1} (a_1 u^2 + a_2 v^2) \mathrm{d}\Gamma. \quad (4)$$

Similar as shown in [14], we give the following assumptions.

(H1) The domain $\Omega$ is of class $C^2$.

(H2) The partition of $\Gamma$ satisfies the condition $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.

(H3) Let $\boldsymbol{H}$ be a vector field on Riemannian manifold $(\mathbf{R}^n, g)$ such that

$$D\boldsymbol{H}(\boldsymbol{X}, \boldsymbol{X}) \geqslant b \, | \boldsymbol{X} |_g^2, \quad \forall \boldsymbol{x} \in \mathbf{R}^n_x, \boldsymbol{x} \in \Omega$$

for some constant $b > 0$, and

$$\boldsymbol{H} \cdot \boldsymbol{\nu} \leqslant 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \inf{}_{\Gamma_1} \boldsymbol{H} \cdot \boldsymbol{\nu} > 0$$

(H4) The coefficients $a_1, a_2$ are nonnegative and they belong to $C^1(\Gamma_1)$. Moreover, either $\Gamma_0 \neq \emptyset$ or $\inf_{\Gamma_1} a_1 > 0$ and $\inf_{\Gamma_1} a_2 > 0$.

(H5) The function $\alpha$ is nonnegative and belongs to $L^\infty(\Omega)$.

(H6) The functions $g_1$ and $g_2$ are continuous, nondecreasing, and

$$g_i(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{x} = 0.$$

Furthermore, there exists a constant $C$ such that

$$| g_i(\boldsymbol{x}) | \leqslant 1 + C | \boldsymbol{x} |, \quad \forall \boldsymbol{x} \in \mathbf{R}.$$

Making use of standard semigroup arguments, we have the following result:

**Theorem 1** Let the above hypotheses (H1)—(H6) be satisfied, then for all initial data $u_0, v_0 \in H^1_{\Gamma_0}(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$, problem (1) has a unique weak solution satisfying

$$u, v \in C(\mathbf{R}_+, H^1_{\Gamma_0}(\Omega)) \cap C^1(\mathbf{R}_+, L^2(\Omega)).$$

Furthermore, if $g_1, g_2$ are globally Lipschitz cotinuous and $u_0, v_0 \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$, $u_1, v_1 \in H^1_{\Gamma_0}(\Omega)$ are such that

$$\frac{\partial u_0}{\partial \nu_{\mathscr{A}}} + a_1 u_0 + g_1(u_1) = 0,$$

and $\dfrac{\partial v_0}{\partial \boldsymbol{\nu}_{\mathscr{A}}} + a_2 v_0 + g_2(v_1) = 0$ on $\Gamma_1$,

then the solution is more regular and

$$u,v \in L^{\infty}(\mathbf{R}_+, H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \cap W^{1,\infty}(\mathbf{R}_+, H^1_{\Gamma_0}(\Omega)) \cap W^{2,\infty}(\mathbf{R}_+, L^2(\Omega)). \quad (5)$$

**Remark 1** The well-posedness and regularity of problem (1) is obtained by the same arguments as in the case of constant coefficients (see [15]) but with a few modifications, we omit the details.

Our main result is the following theorem.

**Theorem 2** In addition to the assumptions (H1)—(H6), we further assume $u_0, v_0 \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$ and $u_1, v_1 \in H^1_{\Gamma_0}(\Omega)$ such that

$$\frac{\partial u_0}{\partial \boldsymbol{\nu}_{\mathscr{A}}} + a_1 u_0 + g_1(u_1) = 0,$$

and $\dfrac{\partial v_0}{\partial \boldsymbol{\nu}_{\mathscr{A}}} + a_2 v_0 + g_2(v_1) = 0$ on $\Gamma_1$.

Assume that there exist $p$ such that

$$p = 1, \quad \text{if } n = 1,$$
$$p > 1, \quad \text{if } n = 2,$$
$$p \geqslant n - 1, \quad \text{if } n \geqslant 3,$$

and two positive constants $C_1, C_2$ such that

$$C_1 |\boldsymbol{x}|^p \leqslant g_i(\boldsymbol{x}) \leqslant C_2 |\boldsymbol{x}|^{1/p}, \quad \text{if } |\boldsymbol{x}| \leqslant 1, \quad (6)$$
$$C_1 |\boldsymbol{x}| \leqslant |g_i(\boldsymbol{x})| \leqslant C_2 |\boldsymbol{x}|, \quad \text{if } |\boldsymbol{x}| \geqslant 1, \quad (7)$$

$(i = 1, 2)$, then the solution of (1) satisfies the estimates:

$$E(t) \leqslant C t^{-\frac{2}{p-1}}, \quad \forall t > 0 \text{ if } p > 1,$$
$$E(t) \leqslant C E(0) e^{-\omega t}, \quad \forall t > 0 \text{ if } p = 1,$$

where $C$ and $\omega$ are positive constants.

We need the following lemma:

**Lemma 1**[2] Let $E: \mathbf{R}_+ \to \mathbf{R}_+$ be a non-increasing function and assume that there exist a nonnegative number $q$ and a positive number $A$ such that

$$\int_t^{\infty} E^{q+1} \mathrm{d}s \leqslant A E(t), \quad \text{for all } t \geqslant 0.$$

Then putting $T = A E(0)^{-q}$, we have

$$E(t) \leqslant E(0) \left( \frac{T + qT}{T + qt} \right)^{1/q}, \quad \text{for all } t \geqslant T$$

if $q > 0$, and

$$E(t) \leqslant E(0) \mathrm{e}^{1 - \frac{t}{T}}, \quad \text{for all } t \geqslant T$$

if $q = 0$.

## 2 Proof of the main theorem

In this section, we will combine the Riemannian geometry method and multiplier technique used in [8, 14] to prove our main result. We first state the following lemmas.

**Lemma 2** $E(t)$ is a non-increasing function for $t \geqslant 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -\int_{\Gamma_1} (g_1(u_t) u_t + g_2(v_t) v_t) \mathrm{d}\Gamma. \quad (8)$$

**Proof** Multiplying the first equation of (1) by $u_t$ and the second equation by $v_t$, integrating them over $\Omega$, adding them together and using the boundary conditions, we obtain

$$E(t) - E(0) = -\int_0^t \int_{\Gamma_1} (g_1(u_t) u_t + g_2(v_t) v_t) \mathrm{d}\Gamma \mathrm{d}s, \quad t \geqslant 0. \quad (9)$$

Note that $\mathscr{A} u = -\mathrm{div}_0 \nabla_g u$, where $\mathrm{div}_0$ denotes the divergence in the Euclidean metric. Then we have the equality (8).

**Lemma 3** Let $(u, v)$ be the solution of (1) and satisfy (5).

1) Assume that $\boldsymbol{H}$ is a vector field on $\overline{\Omega}$. Then

$$\int_S^T E^{\frac{p-1}{2}} \int_{\Omega} \left( D\boldsymbol{H}(\nabla_g u, \nabla_g u) + D\boldsymbol{H}(\nabla_g v, \nabla_g v) \right) \mathrm{d}x \mathrm{d}t + \frac{1}{2} \int_S^T E^{\frac{p-1}{2}} \int_{\Omega} (u_t^2 + v_t^2 - |\nabla_g u|_g^2 - |\nabla_g v|_g^2) \mathrm{div}_0 \boldsymbol{H} \mathrm{d}x \mathrm{d}t - \frac{1}{2} \int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_0} (|\nabla_g u|_g^2 + |\nabla_g v|_g^2) \boldsymbol{H} \cdot \boldsymbol{\nu} \mathrm{d}\Gamma \mathrm{d}t =$$
$$\left[ E^{\frac{p-1}{2}} \int_{\Omega} (u_t \boldsymbol{H}(u) + v_t \boldsymbol{H}(v)) \mathrm{d}x \right]_T^S + \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E_t \int_{\Omega} (u_t \boldsymbol{H}(u) + v_t \boldsymbol{H}(v)) \mathrm{d}x \mathrm{d}t +$$
$$\int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_1} \left[ \frac{1}{2} (u_t^2 + v_t^2 - |\nabla_g u|_g^2 - |\nabla_g v|_g^2) \boldsymbol{H} \cdot v \right] \mathrm{d}\Gamma \mathrm{d}t - \int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_1} [(a_1 u + g_1(u_t)) \boldsymbol{H}(u) + (a_2 v + g_2(v_t)) \boldsymbol{H}(v)] \mathrm{d}\Gamma \mathrm{d}t - \int_S^T E^{\frac{p-1}{2}} \int_{\Omega} (\boldsymbol{H}(u) - \boldsymbol{H}(v)) \alpha (u - v) \mathrm{d}x \mathrm{d}t. \quad (10)$$

2) Let $P = \frac{1}{2}(\mathrm{div}_0 \boldsymbol{H} - b) \in C^2(\overline{\Omega})$. Then

$$\int_S^T E^{\frac{p-1}{2}} \int_{\Omega} (|\nabla_g u|_g^2 + |\nabla_g v|_g^2 - u_t^2 - v_t^2) P \mathrm{d}x \mathrm{d}t = \left[ E^{\frac{p-1}{2}} \int_{\Omega} (u_t P u + v_t P v) \mathrm{d}x \right]_T^S - \int_S^T E^{\frac{p-1}{2}} \int_{\Omega} \alpha (u - v)(Pu - Pv) \mathrm{d}x \mathrm{d}t +$$
$$\frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E_t \int_{\Omega} (u_t P u + v_t P v) \mathrm{d}x \mathrm{d}t - \frac{1}{2} \int_S^T E^{\frac{p-1}{2}} \int_{\Omega} (u^2 + v^2) \mathscr{A} P \mathrm{d}x \mathrm{d}t -$$
$$\int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_1} \left[ \frac{1}{2} (u^2 + v^2) \frac{\partial P}{\partial \boldsymbol{\nu}_{\mathscr{A}}} + \left( a_1 u + g_1(u_t) \right) P u + \left( a_2 v + g_2(v_t) \right) P v \right] \mathrm{d}\Gamma \mathrm{d}t. \quad (11)$$

**Proof**

1) We have

$$0 = \int_S^T E^{\frac{p-1}{2}} \int_{\Omega} \boldsymbol{H}(u)(u_{tt} + \mathscr{A} u + \alpha(u - v)) \mathrm{d}x \mathrm{d}t =$$
$$\left[ E^{\frac{p-1}{2}} \int_{\Omega} u_t \boldsymbol{H}(u) \mathrm{d}x \right]_S^T - \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E_t \int_{\Omega} u_t \boldsymbol{H}(u) \mathrm{d}x \mathrm{d}t - \int_S^T E^{\frac{p-1}{2}} \int_{\Omega} u_t \boldsymbol{H}(u_t) \mathrm{d}x \mathrm{d}t - \int_S^T E^{\frac{p-1}{2}} \int_{\Omega} \boldsymbol{H}(u) \mathrm{div}_0 \nabla_g u \mathrm{d}x \mathrm{d}t +$$

$$\int_S^T E^{\frac{p-1}{2}} \int_\Omega \alpha(u-v)\boldsymbol{H}(u)\,\mathrm{d}x\mathrm{d}t. \tag{12}$$

Integrating by parts, using Green's formula, the identity in [7] (see Lemma 2.1) and the boundary conditions, we have

$$\int_\Omega (u_t\boldsymbol{H}(u_t)+\boldsymbol{H}(u)\mathrm{div}_0\nabla_g u)\,\mathrm{d}x = -\int_\Omega D\boldsymbol{H}(\nabla_g u,\nabla_g u)\mathrm{d}x - \frac{1}{2}\int_\Omega (u_t^2-|\nabla_g u|_g^2)\mathrm{div}_0\boldsymbol{H}\mathrm{d}x + \frac{1}{2}\int_{\Gamma_0}|\nabla_g u|_g^2\boldsymbol{H}\cdot\boldsymbol{\nu}\mathrm{d}\Gamma + \int_{\Gamma_1}\frac{1}{2}(u_t^2-|\nabla_g u|_g^2)\boldsymbol{H}\cdot\boldsymbol{\nu}\mathrm{d}\Gamma - \int_{\Gamma_1}(a_1 u+g_1(u_t))\boldsymbol{H}(u)\mathrm{d}\Gamma. \tag{13}$$

Combining (12) and (13), and noticing that analogous identities hold for $v$. Taking their sum we get (10).

2) Similarly to (i), we obtain

$$0 = \int_S^T E^{\frac{p-1}{2}}\int_\Omega Pu(u_{tt}+\mathscr{A}u+\alpha(u-v))\mathrm{d}x\mathrm{d}t = \left[E^{\frac{p-1}{2}}\int_\Omega u_t Pu\mathrm{d}x\right]_S^T - \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E_t\int_\Omega u_t Pu\mathrm{d}x\mathrm{d}t - \int_S^T E^{\frac{p-1}{2}}\int_\Omega Pu_t^2\mathrm{d}x\mathrm{d}t - \int_S^T E^{\frac{p-1}{2}}\int_\Omega Pu\mathrm{div}_0\nabla_g u\mathrm{d}x\mathrm{d}t + \int_S^T E^{\frac{p-1}{2}}\int_\Omega \alpha(u-v)Pu\mathrm{d}x\mathrm{d}t. \tag{14}$$

An analogous identity holds for $v$. Then summing them, we have

$$\left[E^{\frac{p-1}{2}}\int_\Omega (u_tPu+v_tPv)\mathrm{d}x\right]_T^S - \int_S^T E^{\frac{p-1}{2}}\int_\Omega \alpha(u-v)(Pu-Pv) + \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E_t\int_\Omega (u_tPu+v_tPv)\mathrm{d}x\mathrm{d}t = -\int_S^T E^{\frac{p-1}{2}}\int_\Omega (Pu_t^2+Pv_t^2)\mathrm{d}x\mathrm{d}t - \int_S^T E^{\frac{p-1}{2}}\int_\Omega (Pu\mathrm{div}_0\nabla_g u+Pv\mathrm{div}_0\nabla_g v)\mathrm{d}x\mathrm{d}t. \tag{15}$$

We have

$$\int_\Omega (Pu\mathrm{div}_0\nabla_g u+Pv\mathrm{div}_0\nabla_g v)\mathrm{d}x = -\int_\Omega \Big(P(|\nabla_g u|_g^2+|\nabla_g v|_g^2)+\frac{1}{2}(u^2+v^2)\mathscr{A}P\Big)\mathrm{d}x - \int_{\Gamma_1}\Big[\frac{1}{2}(u^2+v^2)\frac{\partial P}{\partial\boldsymbol{\nu}_{\mathscr{A}}} + (a_1u+g_1(u_t))Pu + (a_2v+g_2(v_t))Pv\Big]\mathrm{d}\Gamma. \tag{16}$$

In the last step we used the boundary conditions and the following identity in [7]

$$\langle\nabla_g u,\nabla_g(Pu)\rangle = P|\nabla_g u|_g^2 + \frac{1}{2}\mathrm{div}_0(u^2\nabla_g P) + \frac{1}{2}u^2\mathscr{A}P.$$

Then, substituting (16) into (15), we have (11).

**Lemma 4** It holds that

$$\int_S^T E^{\frac{p+1}{2}}\mathrm{d}t \leqslant CE^{\frac{p+1}{2}}(S) + C\int_S^T E^{\frac{p-1}{2}}\int_\Omega (u^2+v^2)\mathrm{d}x\mathrm{d}t + C\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(u_t^2+v_t^2+u^2+v^2+g_1(u_t)^2+g_2(v_t)^2)\mathrm{d}\Gamma\mathrm{d}t, \tag{17}$$

for all $0\leqslant S<T<+\infty$.

**Proof** Applying Lemma 3(ii), we have

$$\frac{1}{2}\int_S^T E^{\frac{p-1}{2}}\int_\Omega (|\nabla_g u|_g^2+|\nabla_g v|_g^2-u_t^2-v_t^2)\cdot(\mathrm{div}_0\boldsymbol{H}-b)\mathrm{d}x\mathrm{d}t = \frac{1}{2}\left[E^{\frac{p-1}{2}}\int_\Omega(\mathrm{div}_0\boldsymbol{H}-b)(u_tu+v_tv)\mathrm{d}x\right]_T^S + \frac{p-1}{4}\int_S^T E^{\frac{p-3}{2}}E_t\int_\Omega(\mathrm{div}_0\boldsymbol{H}-b)(u_tu+v_tv)\mathrm{d}x\mathrm{d}t - \frac{1}{4}\int_S^T E^{\frac{p-1}{2}}\int_\Omega(u^2+v^2)\mathscr{A}(\mathrm{div}_0\boldsymbol{H}-b)\mathrm{d}x\mathrm{d}t - \int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}\frac{1}{4}(u^2+v^2)\frac{\partial(\mathrm{div}_0\boldsymbol{H})}{\partial\boldsymbol{\nu}_{\mathscr{A}}}\mathrm{d}\Gamma\mathrm{d}t - \frac{1}{2}\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}[(a_1u+g_1(u_t))(\mathrm{div}_0\boldsymbol{H}u-bu) + (a_2v+g_2(v_t))(\mathrm{div}_0\boldsymbol{H}v-bv)]\mathrm{d}\Gamma\mathrm{d}t - \frac{1}{2}\int_S^T E^{\frac{p-1}{2}}\int_\Omega\alpha(u-v)(\mathrm{div}_0\boldsymbol{H}-b)(u-v)\mathrm{d}x\mathrm{d}t. \tag{18}$$

From (10) and (18), we get

$$\int_S^T E^{\frac{p-1}{2}}\int_\Omega\Big(D\boldsymbol{H}(\nabla_g u,\nabla_g u)+D\boldsymbol{H}(\nabla_g v,\nabla_g v)+\frac{1}{2}b(u_t^2+v_t^2-|\nabla_g u|_g^2-|\nabla_g v|_g^2)\Big)\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_0}(|\nabla_g u|_g^2+|\nabla_g v|_g^2)\boldsymbol{H}\cdot\boldsymbol{\nu}\mathrm{d}\Gamma\mathrm{d}t = \left[E^{\frac{p-1}{2}}\int_\Omega(u_tM(u)+v_tM(v))\mathrm{d}x\right]_T^S + \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E_t\int_\Omega(u_tM(u)+v_tM(v))\mathrm{d}x\mathrm{d}t - \frac{1}{4}\int_S^T E^{\frac{p-1}{2}}\int_\Omega(u^2+v^2)\mathscr{A}(\mathrm{div}_0H-b)\mathrm{d}x\mathrm{d}t + \int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}\Big[\frac{1}{2}(u_t^2+v_t^2-|\nabla_g u|_g^2-|\nabla_g v|_g^2)\boldsymbol{H}\cdot\boldsymbol{\nu} - \frac{1}{4}(u^2+v^2)\frac{\partial(\mathrm{div}_0\boldsymbol{H})}{\partial\boldsymbol{\nu}_{\mathscr{A}}}\Big]\mathrm{d}\Gamma\mathrm{d}t - \int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}[(a_1u+g_1(u_t))M(u) + (a_2v+g_2(v_t))\boldsymbol{M}(v)]\mathrm{d}\Gamma\mathrm{d}t - \int_S^T E^{\frac{p-1}{2}}\int_\Omega\alpha(u-v)(\boldsymbol{M}(u)-\boldsymbol{M}(v))\mathrm{d}x\mathrm{d}t, \tag{19}$$

with $\boldsymbol{M}(S)=\boldsymbol{H}(S)+\frac{1}{2}\mathrm{div}_0\boldsymbol{H}S-\frac{b}{2}S$.

For the left-hand side of (19), we have

$$\int_S^T E^{\frac{p-1}{2}}\int_\Omega\Big(D\boldsymbol{H}(\nabla_g u,\nabla_g u)+D\boldsymbol{H}(\nabla_g v,\nabla_g v)+\frac{1}{2}b(u_t^2+v_t^2-|\nabla_g u|_g^2-|\nabla_g v|_g^2)\Big)\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_0}(|\nabla_g u|_g^2+|\nabla_g v|_g^2)\boldsymbol{H}\cdot\boldsymbol{\nu}\mathrm{d}\Gamma\mathrm{d}t \geqslant \frac{1}{2}b\int_S^T E^{\frac{p-1}{2}}\int_\Omega(u_t^2+v_t^2+|\nabla_g u|_g^2+|\nabla_g v|_g^2)\mathrm{d}x\mathrm{d}t, \tag{20}$$

where we have used the assumption (H3). Similarly to

[8], we easily obtain the following estimates,

$$\left[E^{\frac{p-1}{2}}\int_\Omega (u_t\boldsymbol{M}(u)+v_t\boldsymbol{M}(v))\,\mathrm{d}x\right]_T^S \leqslant CE^{\frac{p+1}{2}}(S),$$

$$\frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E_t\int_\Omega (u_t\boldsymbol{M}(u)+v_t\boldsymbol{M}(v))\,\mathrm{d}x\mathrm{d}t \leqslant$$
$$-C\int_S^T E^{\frac{p-1}{2}}E_t\mathrm{d}t \leqslant CE^{\frac{p+1}{2}}(S),$$

$$-\frac{1}{4}\int_S^T E^{\frac{p-1}{2}}\int_\Omega (u^2+v^2)\mathscr{A}(\mathrm{div}_0\boldsymbol{H}-b)\,\mathrm{d}x\mathrm{d}t \leqslant$$
$$C\int_S^T E^{\frac{p-1}{2}}\int_\Omega (u^2+v^2)\,\mathrm{d}x\mathrm{d}t,$$

$$\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}\Big[\frac{1}{2}(u_t^2+v_t^2-|\nabla_g u|_g^2-|\nabla_g v|_g^2)\boldsymbol{H}\cdot\boldsymbol{\nu}-$$
$$\frac{1}{4}(u^2+v^2)\frac{\partial(\mathrm{div}_0\boldsymbol{H})}{\partial\boldsymbol{\nu}_{\mathscr{A}}}\Big]\mathrm{d}\Gamma\mathrm{d}t-$$
$$\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}[(a_1u+g_1(u_t))\boldsymbol{M}(u)+$$
$$(a_2v+g_2(v_t))\boldsymbol{M}(v)]\,\mathrm{d}\Gamma\mathrm{d}t \leqslant$$
$$C\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(u^2+v^2+u_t^2+v_t^2+g_1(u_t)^2+g_2(v_t)^2)\,\mathrm{d}\Gamma\mathrm{d}t.$$

Then by Cauchy-Schwarz inequality and the boundedness of $\Omega$, we have

$$\left|\int_\Omega (u-v)\boldsymbol{M}(u-v)\,\mathrm{d}x\right| \leqslant CE.$$

Furthermore, we obtain

$$-\int_S^T E^{\frac{p-1}{2}}\int_\Omega \alpha(u-v)(\boldsymbol{M}(u)-\boldsymbol{M}(v))\,\mathrm{d}x\mathrm{d}t=$$
$$-\int_S^T E^{\frac{p-1}{2}}\int_\Omega \alpha(u-v)\boldsymbol{M}(u-v)\,\mathrm{d}x\mathrm{d}t \leqslant C\int_S^T E^{\frac{p+1}{2}}\mathrm{d}t.$$

By (4) and the above estimates, we have

$$\int_S^T E^{\frac{p+1}{2}}\mathrm{d}t \leqslant CE^{\frac{p+1}{2}}(S)+C\int_S^T E^{\frac{p-1}{2}}\int_\Omega (u^2+v^2)\,\mathrm{d}x\mathrm{d}t+$$
$$C\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(u_t^2+v_t^2+u^2+v^2+$$
$$g_1(u_t)^2+g_2(v_t)^2)\,\mathrm{d}\Gamma\mathrm{d}t.$$

The next result presents an inequality where the lower order terms on the right hand side of (17) will be absorbed.

**Lemma 5** It holds that

$$\int_S^T E^{\frac{p-1}{2}}\left(\int_\Omega (u^2+v^2)\,\mathrm{d}x+\int_{\Gamma_1}(u^2+v^2)\,\mathrm{d}\Gamma\right)\mathrm{d}t \leqslant$$
$$C\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(u_t^2+v_t^2+g_1(u_t)^2+g_2(v_t)^2)\,\mathrm{d}\Gamma\mathrm{d}t \quad (21)$$

for all $0\leqslant S<T<+\infty$.

**Proof** By Lemma 2 we get

$$E^{\frac{p+1}{2}}(S)-E^{\frac{p+1}{2}}(T)=-\frac{2}{p+1}\int_S^T E^{\frac{p-1}{2}}E_t\mathrm{d}t \leqslant$$
$$C\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(u_t^2+v_t^2+g_1(u_t)^2+g_2(v_t)^2)\,\mathrm{d}\Gamma\mathrm{d}t. \quad (22)$$

From (17) and (22), we have

$$E^{\frac{p+1}{2}}(S) \leqslant C\int_S^T E^{\frac{p-1}{2}}\int_\Omega (u^2+v^2)\,\mathrm{d}x\mathrm{d}t+$$
$$C(S,T)\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(u^2+v^2+u_t^2+v_t^2+g_1(u_t)^2+$$
$$g_2(v_t)^2)\,\mathrm{d}\Gamma\mathrm{d}t. \quad (23)$$

We will argue by contradiction. Let $(u_m,v_m)$ be a sequence of solution of (1) such that

$$\int_S^T E_m^{\frac{p-1}{2}}\left(\int_\Omega (u_m^2+v_m^2)\,\mathrm{d}x+\int_{\Gamma_1}(u_m^2+v_m^2)\,\mathrm{d}\Gamma\right)\mathrm{d}t=1, \quad (24)$$

$$\lim_{m\to\infty}\int_S^T E_m^{\frac{p-1}{2}}\int_{\Gamma_1}(u_{mt}^2+v_{mt}^2+g_1(u_{mt})^2+$$
$$g_2(v_{mt})^2)\,\mathrm{d}\Gamma\mathrm{d}t=0, \quad (25)$$

where $E_m$ is the energy of $(u_m,v_m)$. From (23), (24) and (25) we see that $E_m(S)$ is bounded. Then, we get subsequence, still denoted by $(u_m,v_m)$, that satisfies the following properties:

$$u_m(S)\to\bar{u}_0,\quad v_m(S)\to\bar{v}_0\quad \text{weakly in } H^1(\Omega),$$
$$u_{mt}(S)\to\bar{u}_1,\quad v_{mt}(S)\to\bar{v}_1\quad \text{weakly in } L^2(\Omega).$$

Setting $(\bar{u},\bar{v})$ is the solution of (1) with the initial data:

$$\bar{u}(x,S)=\bar{u}_0(x),\quad \bar{v}(x,S)=\bar{v}_0(x)\ \text{ in }\Omega$$
$$\bar{u}_t(x,S)=\bar{u}_1(x),\quad \bar{v}_t(x,S)=\bar{v}_1(x)\ \text{ in }\Omega.$$

Then

$$\{u_m,u_{mt}\}\to\{\bar{u},\bar{u}_t\},\quad \{v_m,v_{mt}\}\to\{\bar{v},\bar{v}_t\}$$
$$\text{weakly star in } L^\infty(S,T;H^1(\Omega)\times L^2(\Omega)).$$

Applying compactness results, we deduce that

$$u_m\to\bar{u},\quad v_m\to\bar{v}$$
$$\text{strongly in } L^2\big(S,T;L^2(\Omega)\cap L^2(\Gamma)\big). \quad (26)$$

**Case 1.** $(\bar{u},\bar{v})\neq(0,0)$. We have

$$\lim_{m\to\infty}E_m(S)=E_0(S)>0, \quad (27)$$

where $E_0$ is the energy of $(\bar{u},\bar{v})$. According to (22), (25) and (27), we have

$$\lim_{m\to\infty}E_m(T)>0.$$

From (25), we can obtain that

$u_{mt}\to 0, v_{mt}\to 0$, strongly in $L^2(S,T;L^2(\Gamma_1))$,

$g_1(u_{mt})\to 0, g_2(v_{mt})\to 0$ strongly in $L^2(S,T;L^2(\Gamma_1))$.

Passing to the limit in the equation, we get for $(\bar{u},\bar{v})$,

$$\begin{cases}\bar{u}_{tt}+\mathscr{A}\bar{u}+\alpha(\bar{u}-\bar{v})=0, & \text{in } \Omega\times(S,T),\\ \bar{v}_{tt}+\mathscr{A}\bar{v}+\alpha(\bar{v}-\bar{u})=0, & \text{in } \Omega\times(S,T),\\ \bar{u}=0,\bar{v}=0, & \text{on } \Gamma_0\times(S,T),\\ \dfrac{\partial\bar{u}}{\partial\boldsymbol{\nu}_{\mathscr{A}}}+a_1\bar{u}=0,\bar{u}_t=0 & \text{on } \Gamma_1\times(S,T),\\ \dfrac{\partial\bar{v}}{\partial\boldsymbol{\nu}_{\mathscr{A}}}+a_2\bar{v}=0,\bar{v}_t=0 & \text{on } \Gamma_1\times(S,T),\end{cases}$$

and for $\bar{u}_t=\mu,\bar{v}_t=\nu$,

$$\begin{cases}\mu_{tt}+\mathscr{A}\mu+\alpha(\mu-\nu)=0, & \text{in } \Omega\times(S,T),\\ \nu_{tt}+\mathscr{A}\nu+\alpha(\nu-\mu)=0, & \text{in } \Omega\times(S,T),\\ \mu=0,\nu=0, & \text{on } \Gamma_0\times(S,T),\\ \dfrac{\partial\mu}{\partial\boldsymbol{\nu}_{\mathscr{A}}}=0,\mu=0 & \text{on } \Gamma_1\times(S,T),\\ \dfrac{\partial\nu}{\partial\boldsymbol{\nu}_{\mathscr{A}}}=0,\nu=0, & \text{on } \Gamma_1\times(S,T).\end{cases}$$

It implies that $(\mu,\nu)=(0,0)$ for $T-S$ large enough. Then we have

$$\begin{cases}\mathscr{A}\bar{u}_0=0, & \mathscr{A}\bar{v}_0=0, \quad \text{in } \Omega,\\ \bar{u}_0=0, & \bar{v}_0=0, \quad \text{on } \Gamma_0,\\ \dfrac{\partial \bar{u}_0}{\partial \nu_{\mathscr{A}}}+a_1\bar{u}_0=0, & \quad \text{on } \Gamma_1,\\ \dfrac{\partial \bar{v}_0}{\partial \nu_{\mathscr{A}}}+a_2\bar{v}_0=0, & \quad \text{on } \Gamma_1.\end{cases}$$

with $(\bar{u},\bar{v})=(\bar{u}_0,\bar{v}_0)$. By the assumption (H4), we get $(\bar{u}_0,\bar{v}_0)=(0,0)$. That is $(\bar{u},\bar{v})=(0,0)$, which contradicts our assumption.

**Case 2.** $(\bar{u},\bar{v})=(0,0)$. We conclude by (26) that

$$\lim_{m\to\infty}\int_S^T\left(\int_\Omega(u_m^2+v_m^2)\mathrm{d}x+\int_{\Gamma_1}(u_m^2+v_m^2)\mathrm{d}\Gamma\right)\mathrm{d}t=0. \quad (28)$$

Since $E_m(S)$ is bounded, we have

$$\lim_{m\to\infty}\int_S^T E_m^{\frac{p-1}{2}}\left(\int_\Omega(u_m^2+v_m^2)\mathrm{d}x+\int_{\Gamma_1}(u_m^2+v_m^2)\mathrm{d}\Gamma\right)\mathrm{d}t=0,$$

which contradicts (24). Thus, Lemma 5 is proved.

**Proof of Theorem 2** Combining Lemma 4 and Lemma 5, we obtain

$$\int_S^T E^{\frac{p+1}{2}}dt \leqslant CE^{\frac{p+1}{2}}(S)+$$
$$C\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(u_t^2+v_t^2+g_1(u_t)^2+g_2(v_t)^2)\mathrm{d}\Gamma\mathrm{d}t. \quad (29)$$

Using the growth assumption (6), we have

$$\int_{|u_t|\leqslant 1}(u_t^2+g_1(u_t)^2)\mathrm{d}\Gamma \leqslant$$
$$C\int_{|u_t|\leqslant 1}(u_tg_1(u_t))^{\frac{2}{p+1}}\mathrm{d}\Gamma \leqslant$$
$$C\left(\int_{|u_t|\leqslant 1}(u_tg_1(u_t))\mathrm{d}\Gamma\right)^{\frac{2}{p+1}} \leqslant C|E_t|^{\frac{2}{p+1}}. \quad (30)$$

On the other hand, if $n=1$, we have $u_t\in H^1_{\Gamma_0}(\Omega)\subset L^\infty$ and then

$$\int_{|u_t|\geqslant 1}(u_t^2+g_1(u_t)^2)\mathrm{d}\Gamma \leqslant$$
$$C\int_{|u_t|\geqslant 1}(u_tu_tg_1(u_t))\mathrm{d}\Gamma \leqslant$$
$$C\|u_t\|_{L^\infty}\int_{\Gamma_1}(u_tg_1(u_t))\mathrm{d}\Gamma \leqslant C|E_t|. \quad (31)$$

If $n\geqslant 2$, we get

$$\int_{|u_t|\geqslant 1}(u_t^2+g_1(u_t)^2)\mathrm{d}\Gamma \leqslant$$
$$C\int_{\Gamma_1}|u_t|^{\frac{2p}{p+1}}(u_tg_1(u_t))^{\frac{2}{p+1}}\mathrm{d}\Gamma \leqslant$$
$$C\left\||u_t|^{\frac{2p}{p+1}}\right\|_{(p+1)/(p-1)}\left\|(u_tg_1(u_t))^{\frac{2}{p+1}}\right\|_{(p+1)/2} \leqslant$$
$$C\left\|u_t\right\|_{2p/(p-1)}^{2p/(p+1)}\left\|u_tg_1(u_t)\right\|_1^{2/(p+1)} \leqslant C|E_t|^{2/(p+1)}. \quad (32)$$

where we have used Hölder's inequality and the fact $H^1(\Omega)\subset L^{2p/(p-1)}(\Gamma)$ under the assumption (7).

We have similar inequalities for $v$. Combining (29)—(32), we obtain that

$$\int_S^T E^{\frac{p+1}{2}}\mathrm{d}t \leqslant CE^{\frac{p+1}{2}}(S)+C\int_S^T E^{\frac{p-1}{2}}|E_t|^{\frac{2}{p+1}}\mathrm{d}t.$$

Using the Young inequality, for any fixed $\varepsilon>0$ we have

$$C\int_S^T E^{\frac{p-1}{2}}|E_t|^{\frac{2}{p+1}}\mathrm{d}t \leqslant$$
$$\int_S^T\left(\varepsilon E^{\frac{p+1}{2}}+C(\varepsilon)|E_t|\right)\mathrm{d}t \leqslant$$
$$\varepsilon\int_S^T E^{\frac{p+1}{2}}\mathrm{d}t+C(\varepsilon)E(S).$$

Therefore,

$$(1-\varepsilon)\int_S^T E^{\frac{p+1}{2}}\mathrm{d}t \leqslant CE^{\frac{p+1}{2}}(S)+CE(S),$$

for $0<\varepsilon<1$, whict follows that

$$\int_S^T E^{\frac{p+1}{2}}\mathrm{d}t \leqslant C(E^{\frac{p-1}{2}}(0)+1)E(S),$$

with $p=1$ if $n=1$. Then the proof of Theorem 2 is finished by applying Lemma 1.

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# 变系数紧耦合波动方程组的非线性边界镇定

雷娟[1]
1 南京信息工程大学 数理学院,南京,210044

**摘要** 利用黎曼几何方法和乘子技巧,讨论了带变系数和非线性边界扩散反馈项的紧耦合波动方程组的镇定,得到了解能量的衰减估计.

**关键词** 乘子方法;非线性边界反馈;黎曼流形;能量衰减