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On equivalence of stability criteria for time-delay systems

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Abstract

In recent years, free weighting matrix approach is an important technique to deal with the delay-dependent stability problem for systems with time-varying delay. In this paper, some simplified delay-dependent stability criteria are derived by using a simple integral inequality, in which no free weighting matrix is involved. These presented results turn out to be equivalent to some latest results but include the least number of variables. This feature can largely reduce the computational burden of the obtained LMI conditions.

Key words

delay-dependent; stability; linear matrix inequality (LMI).

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0 Introduction

The delay-dependent stability analysis for time-delay systems keeps attracting researchers for decades (see [1-6]). The aim for delay-dependent stability analysis has been focusing on effective reduction of the conservatism of the stability conditions. Many methods have been provided for this aim, e.g. the model transformation method[7], the inequality bounding technique method[8]. Recently, a free-weighting matrix method is proposed in [9-10], which is very effective to tackle the delay-dependent stability problem. However, the free weighting matrix method often needs to introduce many slack variables in obtaining LMI conditions and thus leads to a significant increase in the computational demand. One natural question is how to simplify the existing stability results using matrix variables as less as possible while maintaining the effectiveness of the stability conditions.

In this paper, we present some simplified delay-dependent stability criteria for three type of time delay. These results are shown to be equivalent to some existing results in [11-13] but with much less variables. This means that our results are more efficient as the computational burden is largely reduced. Numerical examples are given to verify the effectiveness of the proposed criteria.

1 Problem formulation

Given the following system:

$$\begin{cases}\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{A}_d\boldsymbol{x}(t-\mathrm{d}(t));\\ \boldsymbol{x}(t) = \boldsymbol{\Phi}(t), \quad \forall t \in [-h_2, 0].\end{cases} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbf{R}^n$ is the system state vector. $\boldsymbol{A}$ and $\boldsymbol{A}_\mathrm{d}$ are given matrices. The time delay, $\mathrm{d}(t)$, is a time-varying continuous function that satisfies

$$h_1 \leqslant \mathrm{d}(t) \leqslant h_2, \quad \dot{\mathrm{d}}(t) \leqslant \mu, \tag{2}$$

where $0 \leqslant h_1 < h_2$ and μ are constants. Note that h_1 may not be equal to 0. The initial condition, $\boldsymbol{\Phi}(t)$, is a continuous vector valued initial function of $t \in [h_2, 0]$.

The purpose of this paper is to derive some simplified delay-dependent stability criteria and to turn out to be equivalent to some existing results. To this end, the following Lemma is needed.

Lemma 1[14]. For any positive symmetric constant matrix $\boldsymbol{M} \in \mathbf{R}^{n\times n}$, a scalar $\gamma > 0$, vector function $\boldsymbol{\omega}$: $[0, r] \rightarrow \mathbf{R}^n$, such that the integrations concerned are well defined, then

$$\left(\int_0^{\gamma}\boldsymbol{\omega}(s)\mathrm{d}s\right)^{\mathrm{T}}\boldsymbol{M}\left(\int_0^{\gamma}\boldsymbol{\omega}(s)\mathrm{d}s\right)\leqslant\gamma\int_0^{\gamma}\boldsymbol{\omega}^{\mathrm{T}}(s)\boldsymbol{M}\boldsymbol{\omega}(s)\mathrm{d}s. \quad (3)$$

2 Main result

In this section, we will give the stability conditions for three types of time delay. The conditions are obtained by only using inequality (3) and turn out to be equivalent to some existing results but include the least number of variables.

2.1 Interval time-varying delay case ($h_1 \neq 0$, $h_1 \leqslant \mathrm{d}(t) \leqslant h_2$, $\dot{\mathrm{d}}(t) \leqslant \mu$)

Very recently, a less conservative delay-range-dependent stability condition for time delay systems is presented in [13] by introducing some free weighting matrices. we rewrite this as follows.

Lemma 2[13]. Given scalars h_1, h_2, μ and $h_{12} = h_2 - h_1$. Then, for any delay $\mathrm{d}(t)$ satisfying (2), the time-delay system (1) is asymptotically stable if there exist matrices $\boldsymbol{P}>0, \boldsymbol{Q}_i>0\,(i=1,2,3), \boldsymbol{R}_1>0, \boldsymbol{R}_2>0, \boldsymbol{N}_j, \boldsymbol{M}_j$ and $\boldsymbol{S}_j\,(j=1,2)$, such that the following LMI holds:

$$\boldsymbol{\Omega}=\begin{bmatrix}\boldsymbol{\Omega}_{11} & \boldsymbol{\Omega}_{12} & \boldsymbol{M}_1 & -\boldsymbol{S}_1 & h_2\boldsymbol{N}_1 & -h_{12}\boldsymbol{S}_1 & h_{12}\boldsymbol{M}_1 & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{U}\\ * & \boldsymbol{\Omega}_{22} & \boldsymbol{M}_2 & -\boldsymbol{S}_2 & h_2\boldsymbol{N}_2 & -h_{12}\boldsymbol{S}_2 & h_{12}\boldsymbol{M}_2 & \boldsymbol{A}_d^{\mathrm{T}}\boldsymbol{U}\\ * & * & -\boldsymbol{Q}_1 & 0 & 0 & 0 & 0 & 0\\ * & * & * & -\boldsymbol{Q}_2 & 0 & 0 & 0 & 0\\ * & * & * & * & -h_2\boldsymbol{R}_1 & 0 & 0 & 0\\ * & * & * & * & * & -h_{12}\boldsymbol{R}_3 & 0 & 0\\ * & * & * & * & * & * & -h_{12}\boldsymbol{R}_2 & 0\\ * & * & * & * & * & * & * & -\boldsymbol{U}\end{bmatrix}<0. \quad (4)$$

where

$$\boldsymbol{\Omega}_{11} = \boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{N}_1 + \boldsymbol{N}_1^{\mathrm{T}} + \sum_{i=1}^{3}\boldsymbol{Q}_i,$$
$$\boldsymbol{\Omega}_{12} = \boldsymbol{PA}_d + \boldsymbol{N}_2^{\mathrm{T}} - \boldsymbol{N}_1 + \boldsymbol{S}_1 - \boldsymbol{M}_1,$$
$$\boldsymbol{\Omega}_{22} = -\boldsymbol{N}_2^{\mathrm{T}} - \boldsymbol{N}_2 + \boldsymbol{S}_2^{\mathrm{T}} + \boldsymbol{S}_2 - \boldsymbol{M}_2^{\mathrm{T}} - \boldsymbol{M}_2 - (1-\boldsymbol{\mu})\boldsymbol{Q}_3,$$
$$\boldsymbol{R}_3 = \boldsymbol{R}_1 + \boldsymbol{R}_2.$$

and $*$ denotes the symmetric terms in a symmetric matrix.

Lemma 2 is based on the following Lyapunov-Krasovskii functional

$$\begin{aligned}\boldsymbol{V}(t) = {} & \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{x}(t) + \int_{-h_2}^{0}\int_{t+\theta}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}_1\dot{\boldsymbol{x}}(s)\mathrm{d}s\mathrm{d}\theta + \\ & \int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}_2\dot{\boldsymbol{x}}(s)\mathrm{d}s\mathrm{d}\theta + \\ & \sum_{i=1}^{2}\int_{t-h_i}^{t}\boldsymbol{x}^{\mathrm{T}}(s)\boldsymbol{Q}_i\boldsymbol{x}(s)\mathrm{d}s + \\ & \int_{t-\mathrm{d}(t)}^{t}\boldsymbol{x}^{\mathrm{T}}(s)\boldsymbol{Q}_3\boldsymbol{x}(s)\mathrm{d}s \end{aligned} \quad (5)$$

where $\boldsymbol{P}>0, \boldsymbol{R}_1>0, \boldsymbol{R}_2>0, \boldsymbol{Q}_i>0\,(i=1,2,3)$.

We choose the same Lyapunov-Krasovskii functional of form (5). Based on the functional, the following result can be obtained.

Theorem 1. Given scalars h_1, h_2, μ and $h_{12} = h_2 - h_1$. Then, for any delay $\mathrm{d}(t)$ satisfying (2), the time-delay system (1) is asymptotically stable if there exist matrices $\boldsymbol{P}>0, \boldsymbol{Q}_i>0\,(i=1,2,3), \boldsymbol{R}_1>0, \boldsymbol{R}_2>0$, such that the following LMI holds:

$$\boldsymbol{\Gamma}=\begin{bmatrix}\boldsymbol{\Gamma}_{11} & \boldsymbol{PA}_d+\frac{1}{h_2}\boldsymbol{R}_1 & 0 & 0 & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{U}\\ * & \boldsymbol{\Gamma}_{22} & \frac{1}{h_{12}}\boldsymbol{R}_2 & \frac{1}{h_{12}}\boldsymbol{R}_3 & \boldsymbol{A}_d^{\mathrm{T}}\boldsymbol{U}\\ * & * & -\boldsymbol{Q}_1-\frac{1}{h_{12}}\boldsymbol{R}_2 & 0 & 0\\ * & * & * & \boldsymbol{\Gamma}_{44} & 0\\ * & * & * & * & -\boldsymbol{U}\end{bmatrix}<0. \quad (6)$$

where

$$\boldsymbol{\Gamma}_{11} = \boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \sum_{i=1}^{3}\boldsymbol{Q}_i - \frac{1}{h_2}\boldsymbol{R}_1,$$
$$\boldsymbol{\Gamma}_{22} = -(1-\mu)\boldsymbol{Q}_3 - \frac{1}{h_2}\boldsymbol{R}_1 - \frac{1}{h_{12}}(\boldsymbol{R}_1 + 2\boldsymbol{R}_2),$$
$$\boldsymbol{\Gamma}_{44} = -\boldsymbol{Q}_2 - \frac{1}{h_{12}}(\boldsymbol{R}_1 + \boldsymbol{R}_2), \quad \boldsymbol{R}_3 = \boldsymbol{R}_1 + \boldsymbol{R}_2,$$
$$\boldsymbol{U} = h_2\boldsymbol{R}_1 + h_{12}\boldsymbol{R}_2.$$

and $*$ denotes the symmetric terms in a symmetric matrix.

Proof. Calculating the time derivative of $\boldsymbol{V}(t)$ along the solution of (1) yields

$$\begin{aligned}\dot{\boldsymbol{V}}(t) \leqslant {} & 2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}(\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{A}_d\boldsymbol{x}(t-\mathrm{d}(t))) + \\ & h_2\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{R}_1\dot{\boldsymbol{x}}(t) - \int_{t-h_2}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}_1\dot{\boldsymbol{x}}(s)\mathrm{d}s + \\ & h_{12}\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{R}_2\dot{\boldsymbol{x}}(t) - \int_{t-h_2}^{t-h_1}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}_2\dot{\boldsymbol{x}}(s)\mathrm{d}s + \\ & \boldsymbol{x}^{\mathrm{T}}(t)(\boldsymbol{Q}_1 + \boldsymbol{Q}_2 + \boldsymbol{Q}_3)\boldsymbol{x}(t) - \\ & \boldsymbol{x}^{\mathrm{T}}(t-h_1)\boldsymbol{Q}_1\boldsymbol{x}(t-h_1) - \\ & \boldsymbol{x}^{\mathrm{T}}(t-h_2)\boldsymbol{Q}_2\boldsymbol{x}(t-h_2) - \\ & (1-\mu)\boldsymbol{x}^{\mathrm{T}}(t-\mathrm{d}(t))\boldsymbol{Q}_3\boldsymbol{x}(t-\mathrm{d}(t)). \end{aligned} \quad (7)$$

with $h_{12} = h_2 - h_1$. On the other hand, it follows from Lemma 1, we have

$$\begin{aligned} & -\int_{t-h_2}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}_1\dot{\boldsymbol{x}}(s)\mathrm{d}s = \\ & -\int_{t-h_2}^{t-\mathrm{d}(t)}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}_1\dot{\boldsymbol{x}}(s)\mathrm{d}s - \int_{t-\mathrm{d}(t)}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)R_1\dot{\boldsymbol{x}}(s)\mathrm{d}s \leqslant \\ & -\frac{1}{h_{12}}\left(\int_{t-h_2}^{t-\mathrm{d}(t)}\dot{\boldsymbol{x}}(s)\mathrm{d}s\right)^{\mathrm{T}}\boldsymbol{R}_1\left(\int_{t-h_2}^{t-\mathrm{d}(t)}\dot{\boldsymbol{x}}(s)\mathrm{d}s\right) - \\ & \frac{1}{h_2}\left(\int_{t-\mathrm{d}(t)}^{t}\dot{\boldsymbol{x}}(s)\mathrm{d}s\right)^{\mathrm{T}}\boldsymbol{R}_1\left(\int_{t-\mathrm{d}(t)}^{t}\dot{\boldsymbol{x}}(s)\mathrm{d}s\right) = \end{aligned}$$

$$\begin{bmatrix} \boldsymbol{x}(t-\mathrm{d}(t)) \\ \boldsymbol{x}(t-h_2) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\frac{1}{h_{12}}\boldsymbol{R}_1 & \frac{1}{h_{12}}\boldsymbol{R}_1 \\ \frac{1}{h_{12}}\boldsymbol{R}_1 & -\frac{1}{h_{12}}\boldsymbol{R}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t-\mathrm{d}(t)) \\ \boldsymbol{x}(t-h_2) \end{bmatrix} +$$

$$\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t-\mathrm{d}(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\frac{1}{h_2}\boldsymbol{R}_1 & \frac{1}{h_2}\boldsymbol{R}_1 \\ \frac{1}{h_2}\boldsymbol{R}_1 & -\frac{1}{h_2}\boldsymbol{R}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t-\mathrm{d}(t)) \end{bmatrix}. \quad (8)$$

$$-\int_{t-h_2}^{t-h_1} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}_2\dot{\boldsymbol{x}}(s)\,\mathrm{d}s =$$

$$-\int_{t-h_2}^{t-\mathrm{d}(t)} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}_2\dot{\boldsymbol{x}}(s)\,\mathrm{d}s - \int_{t-\mathrm{d}(t)}^{t-h_1} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}_2\dot{\boldsymbol{x}}(s)\,\mathrm{d}s \leqslant$$

$$-\frac{1}{h_{12}}\Big(\int_{t-h_2}^{t-\mathrm{d}(t)} \dot{\boldsymbol{x}}(s)\,\mathrm{d}s\Big)^{\mathrm{T}} \boldsymbol{R}_2 \Big(\int_{t-h_2}^{t-\mathrm{d}(t)} \dot{\boldsymbol{x}}(s)\,\mathrm{d}s\Big) -$$

$$\frac{1}{h_{12}}\Big(\int_{t-\mathrm{d}(t)}^{t-h_1} \dot{\boldsymbol{x}}(s)\,\mathrm{d}s\Big)^{\mathrm{T}} \boldsymbol{R}_2 \Big(\int_{t-\mathrm{d}(t)}^{t-h_1} \dot{\boldsymbol{x}}(s)\,\mathrm{d}s\Big) =$$

$$\begin{bmatrix} \boldsymbol{x}(t-\mathrm{d}(t)) \\ \boldsymbol{x}(t-h_2) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\frac{1}{h_{12}}\boldsymbol{R}_2 & \frac{1}{h_{12}}\boldsymbol{R}_2 \\ \frac{1}{h_{12}}\boldsymbol{R}_2 & -\frac{1}{h_{12}}\boldsymbol{R}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t-\mathrm{d}(t)) \\ \boldsymbol{x}(t-h_2) \end{bmatrix} +$$

$$\begin{bmatrix} \boldsymbol{x}(t-h_1) \\ \boldsymbol{x}(t-\mathrm{d}(t)) \end{bmatrix} \begin{bmatrix} -\frac{1}{h_{12}}\boldsymbol{R}_2 & \frac{1}{h_{12}}\boldsymbol{R}_2 \\ \frac{1}{h_{12}}\boldsymbol{R}_2 & -\frac{1}{h_{12}}\boldsymbol{R}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t-h_1) \\ \boldsymbol{x}(t-\mathrm{d}(t)) \end{bmatrix}. \quad (9)$$

Taking (8) and (9) into $\dot{\boldsymbol{V}}(t)$, we have

$$\dot{\boldsymbol{V}}(t) = \boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{\Gamma}_1\boldsymbol{\eta}(t) + h_2\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{R}_1\dot{\boldsymbol{x}}(t) + h_{12}\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{R}_2\dot{\boldsymbol{x}}(t), \quad (10)$$

with

$$\boldsymbol{\Gamma}_1 = \begin{bmatrix} \boldsymbol{\Gamma}_{11} & \boldsymbol{P}\boldsymbol{A}_d + \frac{1}{h_2}\boldsymbol{R}_1 & 0 & 0 \\ * & \boldsymbol{\Gamma}_{22} & \frac{1}{h_{12}}\boldsymbol{R}_2 & \frac{1}{h_{12}}(\boldsymbol{R}_1 + \boldsymbol{R}_2) \\ * & * & -\boldsymbol{Q}_1 - \frac{1}{h_{12}}\boldsymbol{R}_2 & 0 \\ * & * & * & \boldsymbol{\Gamma}_{44} \end{bmatrix},$$

$$\boldsymbol{\eta}(t) = \begin{bmatrix} \boldsymbol{x}^{\mathrm{T}}(t) & \boldsymbol{x}^{\mathrm{T}}(t-\mathrm{d}(t)) & \boldsymbol{x}^{\mathrm{T}}(t-h_1) & \boldsymbol{x}^{\mathrm{T}}(t-h_2) \end{bmatrix}^{\mathrm{T}}.$$

Applying the Schur complement equivalence to (6) gives $\dot{\boldsymbol{V}}(t) < 0$ by (10). Then, the system (1) is asymptotically stable. Q. E. D

Although Lemma 2 and Theorem 1 are obtained via different methods, they turned out to be equivalent. To show this, we give the following theorem.

Theorem 2. Inequality $\boldsymbol{\Omega} < 0$ in Lemma 2 is feasible if and only if $\boldsymbol{\Gamma} < 0$ in Theorem 1 is feasible.

Proof. Note that $\boldsymbol{\Omega}$ in Lemma 2 can be expressed as

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_1 + \boldsymbol{X}\boldsymbol{B} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{T}} < 0$$

where

$$\boldsymbol{\Omega}_1 = \begin{bmatrix} \bar{\boldsymbol{\Omega}}_{11} & \bar{\boldsymbol{\Omega}}_{12} & 0 & 0 & 0 & 0 & 0 \\ * & \bar{\boldsymbol{\Omega}}_{22} & 0 & 0 & 0 & 0 & 0 \\ * & * & -\boldsymbol{Q}_1 & 0 & 0 & 0 & 0 \\ * & * & * & -\boldsymbol{Q}_2 & 0 & 0 & 0 \\ * & * & * & * & -\frac{1}{h_2}\boldsymbol{R}_1 & 0 & 0 \\ * & * & * & * & * & -\frac{1}{h_{12}}\boldsymbol{R}_3 & 0 \\ * & * & * & * & * & * & -\frac{1}{\boldsymbol{h}_{12}}\boldsymbol{R}_2 \end{bmatrix},$$

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{N}_1 & \boldsymbol{S}_1 & \boldsymbol{M}_1 \\ \boldsymbol{N}_2 & \boldsymbol{S}_2 & \boldsymbol{M}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} & 0 & 0 & -\boldsymbol{I} & 0 & 0 \\ 0 & \boldsymbol{I} & 0 & -\boldsymbol{I} & 0 & -\boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} & \boldsymbol{I} & 0 & 0 & 0 & -\boldsymbol{I} \end{bmatrix}.$$

with

$$\bar{\boldsymbol{\Omega}}_{11} = \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \sum_{i=1}^{3}\boldsymbol{Q}_i + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{U}_1\boldsymbol{A},$$

$$\bar{\boldsymbol{\Omega}}_{12} = \boldsymbol{P}\boldsymbol{A}_d + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{U}_1\boldsymbol{A}_d,$$

$$\bar{\boldsymbol{\Omega}}_{22} = -(1-\mu)\boldsymbol{Q}_3 + \boldsymbol{A}_d^{\mathrm{T}}\boldsymbol{U}_1\boldsymbol{A}_d,$$

$$\boldsymbol{R}_3 = \boldsymbol{R}_1 + \boldsymbol{R}_2.$$

From the Finsler's Lemma, it is readily seen that $\boldsymbol{\Omega} < 0$, if and only if

$$\boldsymbol{N}_B^{\mathrm{T}}\boldsymbol{\Omega}_1\boldsymbol{N}_B < 0,$$

hold, where $\boldsymbol{N}_B$ denotes the full-rank matrix representations of the right annihilator of $\boldsymbol{B}$. Since

$$\boldsymbol{N}_B^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{I} & 0 & 0 & 0 & \boldsymbol{I} & 0 & 0 \\ 0 & \boldsymbol{I} & 0 & 0 & -\boldsymbol{I} & \boldsymbol{I} & -\boldsymbol{I} \\ 0 & 0 & 0 & \boldsymbol{I} & 0 & -\boldsymbol{I} & 0 \\ 0 & 0 & \boldsymbol{I} & 0 & 0 & 0 & \boldsymbol{I} \end{bmatrix},$$

It is easy to see that $\boldsymbol{N}_B^{\mathrm{T}}\boldsymbol{\Omega}_1\boldsymbol{N}_B = \boldsymbol{\Gamma} < 0$ when $\boldsymbol{L}_{11} = \boldsymbol{P}, \boldsymbol{R}_1 = \boldsymbol{Z}_1, \boldsymbol{R}_2 = \boldsymbol{Z}_2$, this complete the proof.

Remark 1. In [13], free weighting matrices $\boldsymbol{N}_j$, $\boldsymbol{M}_j, \boldsymbol{S}_j (j=1,2)$ are introduced in order to establish the relationship among the correlated items, e. g. $\boldsymbol{x}(t)$ and $\boldsymbol{x}(t-\mathrm{d}(t))$ to $\int_{t-\mathrm{d}(t)}^{t}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s$, $\boldsymbol{x}(t-\mathrm{d}(t))$ and $\boldsymbol{x}(t-h_2)$ to $\int_{t-h_2}^{t-\mathrm{d}(t)}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s$, $\boldsymbol{x}(t-h_1)$ and $\boldsymbol{x}(t-\mathrm{d}(t))$ to $\int_{t-\mathrm{d}(t)}^{t-h_1}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s$. However, it can be easily seen that the relationship also can be expressed by $\frac{1}{h_{12}}\boldsymbol{R}_1, \frac{1}{h_2}\boldsymbol{R}_1, \frac{1}{h_{12}}\boldsymbol{R}_2$ in (8), (9). Theorem 1 and Lemma 2 are obtained by employing these two formulations, respectively. In fact, they are equivalent to each other according to Theorem 2. Thus, it is naturally unnecessary to introduce such free weighting matrices in expression of these relationships.

2.2 Fast interval time-varying delay case ($h_1 \neq 0$, $h_1 \leq d(t) \leq h_2$)

Lemma 3[12]. For some given scalars $\tau_a = \frac{h_2 + h_1}{2}$ and $\delta = \frac{h_2 - h_1}{2}$, system (1) is asymptotically stable for any $d(t)$ satisfying $h_1 \leq d(t) \leq h_2$, if there exist matrices $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{R} > 0$, $\boldsymbol{S} > 0$, and $\boldsymbol{N}_i, \boldsymbol{M}_i (i = 1, 2, 3)$ of appropriate dimensions such that the following LMI hold

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} & \boldsymbol{\Phi}_{13} & \tau_a \boldsymbol{M}_1^{\mathrm{T}} & \delta \boldsymbol{N}_1^{\mathrm{T}} \boldsymbol{A}_d \\ * & \boldsymbol{\Phi}_{22} & \boldsymbol{\Phi}_{23} & \tau_a \boldsymbol{M}_2^{\mathrm{T}} & \delta \boldsymbol{N}_2^{\mathrm{T}} \boldsymbol{A}_d \\ * & * & \boldsymbol{\Phi}_{33} & \tau_a \boldsymbol{M}_3^{\mathrm{T}} & \delta \boldsymbol{N}_3^{\mathrm{T}} \boldsymbol{A}_d \\ * & * & * & -\tau_a \boldsymbol{R} & 0 \\ * & * & * & * & -\delta \boldsymbol{S} \end{bmatrix} < 0. \quad (11)$$

$$\boldsymbol{\Phi}_{11} = \boldsymbol{Q} + \boldsymbol{N}_1^{\mathrm{T}} \boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}} \boldsymbol{N}_1 + \boldsymbol{M}_1 + \boldsymbol{M}_1^{\mathrm{T}},$$
$$\boldsymbol{\Phi}_{12} = \boldsymbol{N}_1^{\mathrm{T}} \boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}} \boldsymbol{N}_2 - \boldsymbol{M}_1^{\mathrm{T}} + \boldsymbol{M}_2,$$
$$\boldsymbol{\Phi}_{13} = \boldsymbol{P} - \boldsymbol{N}_1^{\mathrm{T}} + \boldsymbol{A}^{\mathrm{T}} \boldsymbol{N}_3 + \boldsymbol{M}_3,$$
$$\boldsymbol{\Phi}_{22} = -\boldsymbol{Q} + \boldsymbol{N}_2^{\mathrm{T}} \boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}} \boldsymbol{N}_2 - \boldsymbol{M}_2^{\mathrm{T}} - \boldsymbol{M}_2,$$
$$\boldsymbol{\Phi}_{23} = -\boldsymbol{N}_2^{\mathrm{T}} + \boldsymbol{A}_d^{\mathrm{T}} \boldsymbol{N}_2 - \boldsymbol{M}_3,$$
$$\boldsymbol{\Phi}_{33} = \tau_a \boldsymbol{R} + \delta \boldsymbol{S} - \boldsymbol{N}_3^{\mathrm{T}} - \boldsymbol{N}_3.$$

The following Theorem is a simplified version of Lemma 3, which can be obtained by using the same functional as in [12]. The proof can follow a similar line as Theorem 1 and hence it is omitted.

Theorem 3. For some given scalars $\tau_a = \frac{h_2 + h_1}{2}$ and $\delta = \frac{h_2 - h_1}{2}$, system (1) is asymptotically stable for any $d(t)$ satisfying $h_1 \leq d(t) \leq h_2$, if there exist matrices $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{R} > 0$ and $\boldsymbol{S} > 0$, such that the following LMI hold

$$\boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{\Delta}_{11} & \frac{1}{\tau_a} \boldsymbol{R} & \boldsymbol{P} \boldsymbol{A}_d & \boldsymbol{A}^{\mathrm{T}} \boldsymbol{\Delta}_{13} \\ * & \boldsymbol{\Delta}_{12} & \frac{1}{\delta} \boldsymbol{S} & 0 \\ * & * & -\frac{1}{\delta} \boldsymbol{S} & \boldsymbol{A}_d^{\mathrm{T}} \boldsymbol{\Delta}_{13} \\ * & * & * & -\boldsymbol{\Delta}_{13} \end{bmatrix} < 0. \quad (12)$$

$$\boldsymbol{\Delta}_{11} = \boldsymbol{Q} + \boldsymbol{A}^{\mathrm{T}} \boldsymbol{P} + \boldsymbol{P} \boldsymbol{A} - \frac{1}{\tau_a} \boldsymbol{R},$$
$$\boldsymbol{\Delta}_{12} = -\boldsymbol{Q} - \frac{1}{\tau_a} \boldsymbol{R} - \frac{1}{\delta} \boldsymbol{S},$$
$$\boldsymbol{\Delta}_{13} = \tau_a \boldsymbol{R} + \delta \boldsymbol{S}.$$

The equivalence between Theorem 3 and Lemma 3 is stated in the following theorem.

Theorem 4. Inequality $\boldsymbol{\Phi} < 0$ in Lemma 3 is feasible if and only if $\boldsymbol{\Delta} < 0$ in Theorem 3 is feasible.

Proof. Rewrite $\boldsymbol{\Phi}$ in Lemma 3 as

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}_1 + \boldsymbol{X}_1 \boldsymbol{B}_1 + \boldsymbol{B}_1^{\mathrm{T}} \boldsymbol{X}_1^{\mathrm{T}} < 0.$$

where

$$\boldsymbol{\Phi}_1 = \begin{bmatrix} \boldsymbol{Q} & 0 & \boldsymbol{P} & 0 & 0 \\ * & -\boldsymbol{Q} & 0 & 0 & 0 \\ * & * & \tau_a \boldsymbol{R} + \delta \boldsymbol{S} & 0 & 0 \\ * & * & * & -\frac{1}{\tau_a} \boldsymbol{R} & 0 \\ * & * & * & * & -\frac{1}{\delta} \boldsymbol{S} \end{bmatrix},$$

$$\boldsymbol{X}_1 = \begin{bmatrix} \boldsymbol{N}_1^{\mathrm{T}} & \boldsymbol{M}_1^{\mathrm{T}} \\ \boldsymbol{N}_2^{\mathrm{T}} & \boldsymbol{M}_2^{\mathrm{T}} \\ \boldsymbol{N}_3^{\mathrm{T}} & \boldsymbol{M}_3^{\mathrm{T}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\boldsymbol{B}_1 = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{A}_d & -\boldsymbol{I} & 0 & \boldsymbol{A}_d \\ \boldsymbol{I} & -\boldsymbol{I} & 0 & -\boldsymbol{I} & 0 \end{bmatrix}.$$

Similar to Theorem 2, explicit null space based calculations yield

$$N\left(\begin{bmatrix} \boldsymbol{A} & \boldsymbol{A}_d & -\boldsymbol{I} & 0 & \boldsymbol{A}_d \\ \boldsymbol{I} & -\boldsymbol{I} & 0 & -\boldsymbol{I} & 0 \end{bmatrix} \right) = \begin{bmatrix} \boldsymbol{I} & 0 & 0 \\ 0 & \boldsymbol{I} & 0 \\ \boldsymbol{A} & 0 & \boldsymbol{A}_d \\ \boldsymbol{I} & -\boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} & \boldsymbol{I} \end{bmatrix}.$$

Then one has

$$\begin{bmatrix} \boldsymbol{I} & 0 & \boldsymbol{A}^{\mathrm{T}} & \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} & 0 & -\boldsymbol{I} & -\boldsymbol{I} \\ 0 & 0 & \boldsymbol{A}_d^{\mathrm{T}} & 0 & \boldsymbol{I} \end{bmatrix} \boldsymbol{\Phi}_1 \begin{bmatrix} \boldsymbol{I} & 0 & 0 \\ 0 & \boldsymbol{I} & 0 \\ \boldsymbol{A} & 0 & \boldsymbol{A}_d \\ \boldsymbol{I} & -\boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} & \boldsymbol{I} \end{bmatrix} =$$

$$\begin{bmatrix} \boldsymbol{\Delta}_{11} + \boldsymbol{A}^{\mathrm{T}} (\tau_a \boldsymbol{R} + \delta \boldsymbol{S}) \boldsymbol{A} & \frac{1}{\tau_a} \boldsymbol{R} & \boldsymbol{P} \boldsymbol{A}_d + \boldsymbol{A}^{\mathrm{T}} (\tau_a \boldsymbol{R} + \delta \boldsymbol{S}) \boldsymbol{A}_d \\ * & \boldsymbol{\Delta}_{12} & \frac{1}{\delta} \boldsymbol{S} \\ * & * & -\frac{1}{\delta} \boldsymbol{S} + \boldsymbol{A}_d^{\mathrm{T}} (\tau_a \boldsymbol{R} + \delta \boldsymbol{S}) \boldsymbol{A}_d \end{bmatrix}. \quad (13)$$

After a simple rearrangement, (13) yields (12). Thus, the equivalence between (12) and (11) is obtained by using Finsler's Lemma.

Remark 2. From the proof of Theorem 4, it is clear that Theorem 3 is equivalent to the result in [12]. This means that the free weighting matrices $\boldsymbol{N}_i, \boldsymbol{M}_i (i = 1, 2, 3)$ in [12] can be removed while maintaining the effectiveness of the stability condition.

2.3 Constant delay case ($h_1 = 0$, $d(t)$ is constant)

For the special case of $h_1 = 0$ and $d(t)$ is constant delay, the following delay-dependent stability criterion can be obtained by setting $\boldsymbol{Q}_1=0$, $\boldsymbol{R}_2=0$, $\boldsymbol{Q}_3=0$ in Theorem 1.

Lemma 5. Given scalars $h_1=0$ and h_2. Then, the time-delay system (1) is asymptotically stable if there exist matrices $\boldsymbol{P}>0$, $\boldsymbol{Q}_2>0$, $\boldsymbol{R}_1>0$, such that the following LMI hold:

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}-\frac{1}{h_2}\boldsymbol{R}_1 & \boldsymbol{PA}_d+\frac{1}{h_2}\boldsymbol{R}_1 & h_1\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}_1 \\ * & -\boldsymbol{Q}_2-\frac{1}{h_2}\boldsymbol{R}_1 & h_2\boldsymbol{A}_d^{\mathrm{T}}\boldsymbol{R}_1 \\ * & * & -h_2\boldsymbol{R}_1 \end{bmatrix}<0. \tag{14}$$

Next we show that Theorem 5 is equivalent to the result in [11]. For this purpose, we list the result in [11] below.

Lemma 4[11]. Given scalars $h_1=0$ and h_2. Then, the time-delay system (1) is asymptotically stable if there exist matrices $\boldsymbol{P}>0$, $\boldsymbol{Q}_2>0$, $\boldsymbol{R}_1>0$ and matrices $\boldsymbol{Y}$, $\boldsymbol{W}$, such that the following LMI holds:

$$\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}_2+\boldsymbol{Y}+\boldsymbol{Y}^{\mathrm{T}} & \boldsymbol{PA}_d-\boldsymbol{Y}+\boldsymbol{W}^{\mathrm{T}} & -h_2\boldsymbol{Y} & h_2\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}_1 \\ * & -\boldsymbol{Q}_2-\boldsymbol{W}-\boldsymbol{W}^{\mathrm{T}} & -h_2\boldsymbol{W} & h_2\boldsymbol{A}_d^{\mathrm{T}}\boldsymbol{R}_1 \\ * & * & -h_2\boldsymbol{R}_1 & 0 \\ * & * & * & -h_2\boldsymbol{R}_1 \end{bmatrix}<0. \tag{15}$$

It is easy to show the equivalence between Theorem 5 and Lemma 4 by using Theorem 2 when

$$\boldsymbol{\Omega}_1=\begin{bmatrix} \boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}_2 & \boldsymbol{PA}_d & 0 & h_2\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}_1 \\ * & -\boldsymbol{Q}_2 & 0 & h_2\boldsymbol{A}_d^{\mathrm{T}}\boldsymbol{R}_1 \\ * & * & -\frac{1}{h_2}\boldsymbol{R}_1 & 0 \\ * & * & * & -h_2\boldsymbol{R}_1 \end{bmatrix},$$

$$\boldsymbol{X}=\begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{W} \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{B}[\boldsymbol{I} \quad -\boldsymbol{I} \quad -\boldsymbol{I} \quad 0].$$

Remark 3. Recently, six stability conditions turn out to be equivalent to Lemma 4 in [15]. It is true that Lemma 4 is more efficient than those in [7,16-19], and [20] since it involves the least number of variables while providing an equivalent stability condition. However, it is worth pointing out that Theorem 5 is also equivalent to Lemma 4 and includes less number of variables.

3 Numerical Example

Now, we provide an example to show the effectiveness of our results.

Example. Consider the time-delay system

$$\dot{\boldsymbol{x}}(t)=\begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}\boldsymbol{x}(t)+\begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}\boldsymbol{x}(t-\mathrm{d}(t)).$$

Table Ⅰ, Table Ⅱ, Table Ⅲ provide some comparisons of the maximum allowable delay bounds and the numbers of the variables involved in the results of this paper and [7,11-13,16-20].

Table Ⅰ Comparisons of maximum allowed delay h_2 and the number of the variables for interval time-varying delay case

h_1	Method	$\mu=0.5$	$\mu=0.9$	number of the variables
1	h_2 by He et. al[13]	2.07	1.74	42
	h_2 by Theroem 1	2.07	1.74	18

Table Ⅱ Comparisons of maximum allowed delay h_2 and the number of the variables for fast interval time-varying delay case

h_1	Method	unknown μ	number of the variables
1	h_2 by Jiang et. al[12]	1.67	36
	h_2 by Theroem 3	1.67	12

Table Ⅲ Comparisons of maximum allowed delay h_2 and the number of the variables for constant delay case

h_1	Method	maximum allowed delay h_2	number of the variables
0	Suplin et. al[16]	4.47	39
	Fridman et. al[7]	4.47	35
	Lee et. al[18]	4.47	35
	Jing et. al[17]	4.47	35
	Wu et. al[19]	4.47	27
	Xu et. al[20]	4.47	25
	Xu et. al[11]	4.47	17
	Theorem 5	4.47	9

From the above Tables, it is seen that our results contain the least number of computed variables while maintaining the effectiveness of the stability conditions. This implies that our method requires less computational power.

4 Conclusions

This paper has established simplified delay-dependent stability criteria for linear systems with time delay. The results are shown to be equivalent to some existing results but include much less variables in the LMI conditions. Thus, the presented method could largely reduce the computational burden in solving LMIs. Both theoretical and numerical comparisons have been provided to show the effectiveness and efficiency of the presentd method.

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时滞系统稳定判据的等价性证明

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摘要　近期,自由权矩阵技术成为处理时滞系统时滞相关稳定的重要方法.论文采用积分不等式方法获得一类不包含任何自由权矩阵的时滞相关稳定简化型判据,这些判据理论上证明是等价于目前存在的一些包含自由权矩阵稳定结论的,因此,通过此法可以有效降低结论的运算量.
关键词　时滞相关;稳定;线性矩阵不等式